A more rigorous derivation of the linearized equation of the thermal explosion previously obtained in [1] is given. By means of this equation the determination of the conditions for ignition of a reacting system involving conductive and convective heat transfer may be considerably simplified. The method of small perturbations is used to examine the stability of solutions of the steady-state equation of thermal explosion [2, 3] for boundary conditions of the third kind.

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It is known [2, 3] that, within the limits of the steady-state approximation, thermal explosion theory leads, in the general case, to solution of the equation

$$\frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial y^2} + \frac{\partial^2 \theta}{\partial z^2} + \delta \exp \theta = 0$$

with the condition

$$\left( \frac{\partial \theta}{\partial n} + \gamma \theta \right)_{|\partial} = 0.$$  (2)

Boundary problem (1), (2) ceases to have a real solution at some $\delta = \delta_\omega$ [2, 3, 5]. The value $\delta = \delta_\omega$ is the critical value, i.e., that at which ignition of the fuel mixture occurs. It is known [3, 5, 6] that when $\delta < \delta_\omega$ several solutions exist for boundary problem (1), (2), while at $\delta = \delta_\omega$ all the solutions merge, and the problem has a unique solution, i.e., the value $\delta = \delta_\omega$ is a branch point of the problem. Using the results of [7], it is easy to find the linear boundary problem

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} + \delta \exp \theta_0 v = 0,$$

$$\left( \frac{\partial v}{\partial n} + \gamma v \right)_{|\partial} = 0,$$

whose eigenvalues coincide with the branch points of the nonlinear boundary problem (1), (2). In spite of the fact that Eq. (3) contains an unknown quantity $\theta_0(x, y, z, \delta_\omega)$, the linearized boundary problem (3), (4) considerably simplifies the determination of $\delta_\omega$ and may serve as a source of additional information. Thus, it was shown, with reference to many examples, in [6] that $\exp \theta_0 = 2.71 \ldots$ on the average, and $\delta_\omega$ may be determined with sufficient accuracy as the first eigenvalue of the boundary problem (3), (4) when $\theta_0(x, y, z, \delta_\omega) = 1$. In the two-dimensional case Eq. (3) coincides in form with the equation of vibration of a diaphragm [8], if we put

$$\exp \theta_0(x, y, \delta_\omega) = \rho(x, y) / F.$$  (5)

If $\theta_0$ has been determined, say, by experiment, $\delta_\omega$ may be determined using the diaphragm analogy, if one takes (5) into account and bears in mind that the boundary conditions correspond to an elastically supported diaphragm edge. Within the limits of the approximation $\theta_0 = 1$ [1, 6, 9], the determination of $\delta_\omega$ using the diaphragm analogy is particularly simple. Upper and lower boundaries may also be determined, using the external properties of the eigenvalues of boundary problem (3), (4). Substituting $\theta_1 < \theta_\omega$ for example, into (3) instead of $\theta_\omega$ we obtain, according to [10, 11], a value of $\delta_\omega$ that we know to be too high, and vice versa. The value of $\theta_0(x, y)$ is easily determined from Eq. (1), by replacing $\exp \theta$ with a value, equal to 1, known to be smaller, while $\theta_2 > \theta_\omega$ may be found from Eq. (1) by replacing $\exp \theta_0$ with the definitely larger value $\exp \theta_2$. Note that the method of successive approximations allow one to construct a sequence of upper and lower functions converging to $\theta_\omega$ which we may then use to construct a sequence of upper and lower numbers converging to $\delta_\omega$. According to [7], the determination of $\delta_\omega$ using an eigenvalue of boundary problem (3), (4) is necessary. The sufficiency of this determination for simple forms of reaction vessel (plane, cylindrical, and spherical) will be shown below.

*Note that the approximation $\exp \theta_0 = e = 2.71 \ldots$ was obtained in [9] from other consideration, but no consideration was given to the accuracy of the value of $\delta_\omega$ obtained.
Let us examine the unsteady equation of thermal conduction with distributed heat sources

\[
\frac{\partial \theta}{\partial t} = \frac{\partial^2 \theta}{\partial x^2} + \frac{k}{x} \frac{\partial \theta}{\partial x} + \delta \exp \theta
\]  

(6)

and boundary conditions

\[
\frac{\partial \theta}{\partial x} \bigg|_{x=0} = 0, \quad \left( \frac{\partial \theta}{\partial x} + \gamma \theta \right) \bigg|_{x=1} = 0.
\]

(7)

We assume that the solution of boundary problem (6), (7) does not differ much from the solution of the corresponding steady-state boundary problem \( \theta(x, t) = \theta_{st}(x) + u(x, t) \).

Substituting in the equation and discarding small quantities of second order and above, we obtain an equation for the perturbation \( u(x, t) \):

\[
\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{k}{x} \frac{\partial u}{\partial x} + \delta \exp \theta_{st}(x) u
\]

(8)

with boundary conditions analogous to (7). We solve problem (8), (7) by separating the variables, putting \( u = v(x) \cdot \exp (-\mu t) \) and substituting it in (8). We obtain an equation for \( v(x) \):

\[
\frac{dv}{dx} + \frac{k}{x} \frac{dv}{dx} + [u + \delta \exp \theta_{st}(x)] v = 0
\]

(9)

with boundary conditions analogous to (7). Thus, the problem of stability is reduced to that of determining the sign of the smallest eigenvalue of boundary problem (9), (7). If \( \mu_0 > 0 \), any initial temperature distribution is gradually dissipated. At \( \mu_0 = 0 \) this is no longer true, and at \( \mu_0 < 0 \) any initial temperature distribution steadily builds up, and an explosion occurs. Thus, \( \mu_0 = 0 \) is a limiting condition for ignition of the fuel mixture. We shall show that if \( \theta_{st}(x) \) is the critical temperature profile in the steady-state theory of thermal explosion [2, 3], and \( \delta \) is the critical value from the viewpoint of this theory, then \( \mu_0 = 0 \).

For a plane vessel the solution of boundary problem (1), (2) has the form

\[
\theta_{st}(x) = \theta_0 - 2 \ln \text{ch} sx.
\]

(10)

A solution of (10) exists, if \( \delta \) is determined from the expression

\[
\delta = 2s^2 \left( 1 - \text{th}^2 s \right) \exp \left( -\frac{2}{\gamma} \text{th} s \right).
\]

(10')

At a certain critical value of \( s = s_c \), \( \delta \) has a maximum. The values \( s_c \) may be found from the equation

\[
s_c \text{th} s_c + s_c^2 \left( 1 - \text{th}^2 s_c \right) + \gamma \left( s_c \text{th} s_c - 1 \right) = 0.
\]

(10'')

This equation has a single positive root, which increases as \( \gamma \) increases. For a plane vessel when \( \delta < \delta_c \) boundary problem (1), (2) has two solutions. At \( s < s_c \) we obtain a first solution for which \( \theta_0 < \theta_{st} \), and at \( s > s_c \) there is a second solution for which \( \theta_0 > \theta_{st} \). For a cylindrical vessel the solution of the steady-state problem (1), (2) has the form [12]:

\[
\theta_{st}(x) = \theta_0 - 2 \ln (1 + mx^2).
\]

(11)

A solution of (11) exists if

\[
\delta = \frac{8s}{(1 + m)^2} \exp \left[ -\frac{4s^2}{\gamma (1 + m)} \right].
\]

(11')

The value \( m = m_s \) corresponding to the maximum \( \delta = \delta_s \) is given by

\[
m_s = \frac{2}{\gamma} \left[ \left( 1 - \frac{4}{\gamma} \right)^{1/2} - 1 \right].
\]

(12)

In this case the first and second solutions are similarly determined. For a spherical vessel the solution of problem (1), (2) may be written in the form [4]:

\[
\theta_{st}(x) = \theta_0 - 2 \int_0^x \frac{y^2 \exp \{ \psi(y) \} dy}{\delta x} dx.
\]

(13)