TWO-DIMENSIONAL MIXED BOUNDARY VALUE PROBLEM IN HEAT AND MASS TRANSFER FOR A CHARACTERISTIC EQUATION WITH MULTIPLE ROOTS

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A system of two differential equations of parabolic type is examined. A boundary value problem is set up and solved. A system of integrodifferential equations is obtained for determining the unknown functions. A method of reducing this to a system of ordinary Volterra integral equations is given.

Consider the system of differential equations

$$\frac{\partial U_i}{\partial t} = \sum_{k=1}^{2} a_{ik} \left( \frac{\partial^2 U_k}{\partial x^2} + \frac{\partial U_k}{\partial y^2} \right) \quad (i=1,2).$$

(1)

We impose the following conditions on the real coefficients $a_{ik}$:

$$(a_{11} + a_{22})^2 - 4(a_{11}a_{22} - a_{12}a_{21}) = 0, \quad a_{11} + a_{22} > 0.$$ 

(2)

If (2) is satisfied, the roots of the characteristic equation

$$\left| \begin{array}{cc} a_{11} - \lambda & a_{12} \\ a_{21} & a_{22} - \lambda \end{array} \right| = 0$$

will be positive and multiple, i.e., $\lambda_1 = \lambda_2 = \lambda > 0$.

With these assumptions we shall solve the following boundary problem.

Problem: To find a solution of the system of equations (1) in the region $\Pi: t > 0, 0 < x < l, -\infty < y < +\infty$, satisfying the initial condition

$$U_i(x, y, t) \bigg|_{t=0} = 0 \quad (i=1,2)$$

(4)

and the boundary conditions

$$(a_{11}^{(1)} U_1 + a_{22}^{(1)} U_2) \bigg|_{x=0} = \psi_1(y, t), \quad \left( \frac{\partial U_1}{\partial x} + h_1 U_1 \right) \bigg|_{x=0} = \psi_2(y, t),$$

$$(a_{11}^{(2)} U_1 + a_{22}^{(2)} U_2) \bigg|_{x=l} = \psi_3(y, t), \quad \left( \frac{\partial U_2}{\partial x} + h_2 U_2 \right) \bigg|_{x=l} = \psi_4(y, t),$$

(5)

where $a_{ij}^{(k)}, h_j (j, k = 1, 2)$ are given constants; $\psi_i(y, t) (i = 1, 2, 3, 4)$ are known continuous finite functions having continuous finite partial derivatives of sufficiently high order, and $\psi_i(y, 0) = 0$. This problem was solved in [3] for the case when the roots of the characteristic equation (2) are positive and different.

We shall seek a solution in the following form [4, 5]:

$$U_i(x, y, t) = \sum_{k, j=1}^{2} B_{ij}^{k} g^{(1)} \ast \omega_{1k}[x, y, t] - \sum_{k=1}^{2} B_{ij}^{k} g^{(2)} \ast \omega_{1k}[x, y, t] +$$

$$+ \sum_{k, j=1}^{2} B_{ij}^{k} g^{(1)} \ast \omega_{2k}[l-x, y, t] - \sum_{k=1}^{2} B_{ij}^{k} g^{(2)} \ast \omega_{2k}[l-x, y, t],$$

(6)

where

$$g^{(1)}(x, y, t) = \frac{1}{2\pi\lambda t} \exp \left[ -\frac{x^2 + y^2}{4\lambda t} \right], \quad g^{(1)}_x = \frac{\partial}{\partial x} g^{(1)}, \quad g^{(1)}_{xx} = \frac{\partial^2}{\partial x^2} g^{(1)};$$

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\[ g^{(2)}(x, y, t) = \frac{x^2 + y^2}{8\pi \lambda^2 t^4} \exp \left[ -\frac{x^2 + y^2}{4\lambda t} \right], \quad g_x = \frac{\partial}{\partial x} g^{(2)}, \quad g_{xx} = \frac{\partial^2}{\partial x^2} g^{(2)}; \]

\[ g \cdot \omega [x, y, t] = \frac{1}{t} \int_{-\infty}^{t} g(x, y - \eta, t - \tau) \omega(\eta, \tau) d \eta. \]

The coefficients \( b_{1j}^k \) in (6) are defined as follows [4]:

\[
B_{11} = a_{11}, \quad B_{12} = \lambda - a_{11}, \quad B_{21} = a_{12}, \quad B_{22} = -a_{12},
\]

\[
B_{31} = a_{21}, \quad B_{32} = -a_{21}, \quad B_{42} = a_{22}, \quad B_{44} = \lambda - a_{22}.
\]

It is then easy to verify that the functions \( U_i (x, y, t) \) given by (6) satisfy (1) and (4).

The unknown functions \( \omega_{ik} (y, t) \) \((i, k = 1, 2)\) in (6) must be defined so that functions \( U_i (x, y, t) \) still satisfy boundary conditions (5). For this purpose functions (6) must be substituted in (5).

We first point out the following lemmas:

**Lemma 1:** If the function \( \omega(y, t) \) has a finite derivative \( \partial \omega/\partial t \) and \( \omega(y, 0) = 0 \), then

\[
\lim_{x \to 0} \frac{\partial}{\partial x^2} \int_{-\infty}^{t} \int_{-\infty}^{\infty} \frac{\omega(\eta, \tau) [x^2 + (y - \eta)^2]}{8\pi \lambda^2 (t - \tau)^2} \exp \left[ -\frac{x^2 + (y - \eta)^2}{4\lambda (t - \tau)} \right] d\eta d\tau = \frac{1}{2\lambda} \frac{\partial \omega}{\partial t}[0, y, t].
\]

**Proof:** We first transform the integral under the derivative sign in (8) as follows:

\[
J = \int_{0}^{t} \int_{-\infty}^{\infty} \frac{\omega(\eta, \tau) [x^2 + (y - \eta)^2]}{8\pi \lambda^2 (t - \tau)^2} \exp \left[ -\frac{x^2 + (y - \eta)^2}{4\lambda (t - \tau)} \right] d\eta d\tau = \int_{0}^{t} d\eta \int_{-\infty}^{\infty} \frac{\partial}{\partial \tau} \omega(\eta, \tau) \frac{1}{2\pi \lambda} \exp \left[ -\frac{x^2 + (y - \eta)^2}{4\lambda (t - \tau)} \right] d\tau.
\]

Integrating the inner integral on the right side of (9) by parts, we obtain

\[
J = \frac{1}{2\pi \lambda} \left\{ \int_{-\infty}^{\infty} \frac{\partial}{\partial \tau} \omega(\eta, \tau) \exp \left[ -\frac{x^2 + (y - \eta)^2}{4\lambda (t - \tau)} \right] d\eta \right\}.
\]

Then

\[
\lim_{x \to 0} \frac{\partial}{\partial x^2} J = \lim_{x \to 0} \frac{1}{2\pi \lambda} \left\{ \int_{-\infty}^{\infty} \frac{1}{2\pi \lambda} \exp \left[ -\frac{x^2 + (y - \eta)^2}{4\lambda (t - \tau)} \right] d\eta \right\} = \lim_{x \to 0} \int_{-\infty}^{t} \int_{-\infty}^{\infty} \frac{1}{2\pi \lambda} \frac{\partial}{\partial \tau} \omega(\eta, \tau) \exp \left[ -\frac{x^2 + (y - \eta)^2}{4\lambda (t - \tau)} \right] d\eta.
\]

Since

\[
\lim_{x \to 0} \int_{-\infty}^{t} \int_{-\infty}^{\infty} \frac{\partial}{\partial \tau} \omega(\eta, \tau) \exp \left[ -\frac{x^2 + (y - \eta)^2}{4\lambda (t - \tau)} \right] d\eta = 0,
\]

We have

\[
\lim_{x \to 0} \frac{\partial^2}{\partial x^2} J = \frac{1}{2\pi \lambda} \int_{0}^{t} \int_{-\infty}^{\infty} g^{(1)}(0, y - \eta, t - \tau) \frac{\partial \omega(\eta, \tau)}{\partial \tau} d\eta d\tau.
\]