TEMPERATURE FIELD OF AN INFINITE PLATE IN THE CASE OF A VARIABLE HEAT-EXCHANGE COEFFICIENT

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UDC 536.12

An infinite system of ordinary differential equations is obtained; account of a finite number of these equations yields an approximate solution for the problem.

We are to solve the one-dimensional heat-conduction equation

\[ \frac{\partial \theta (X, F_0)}{\partial F_0} = \frac{\partial^2 \theta (X, F_0)}{\partial X^2} \]  

with the initial conditions

\[ \theta (X, 0) = f (X) \]  

and the boundary conditions

\[ \frac{\partial \theta (1, F_0)}{\partial X} = Bi (F_0) [\theta_{1m} (F_0) - \theta (1, F_0)], \]  

\[ \frac{\partial \theta (0, F_0)}{\partial X} = 0. \]  

We use the notation

\[ q (F_0) = Bi (F_0) [\theta_{1m} (F_0) - \theta (1, F_0)] \]  

and rewrite condition (3) as

\[ \frac{\partial \theta (1, F_0)}{\partial X} = q (F_0). \]  

Now solving Eq. (1) with boundary conditions (2) and (4) and boundary condition (6) of the second kind, we can find the relation between \( q (F_0) \) and \( \theta (1, F_0) \), which, along with (5), gives the solution of problems (1)-(4).

The solution of problems (1), (2), (4), and (6) can be written [1]

\[ \theta (X, F_0) = \frac{z_0 (F_0)}{2} + \sum_{n=1}^{\infty} (-1)^n \cos n\pi x z_n (F_0), \]  

where

\[ z_0 (F_0) = \frac{1}{\theta} f (X) dX + \int_0^{F_0} q (\tau) d\tau, \]  

\[ z_n (F_0) = \left\{ (-1)^n 2 \int_0^1 f (X) \cos n\pi x dX + 2 \int_0^{F_0} q (\tau) \exp (n\pi)^3 \exp \left[ -(n\pi)^2 F_0 \right] \right\} \]  

\[ n = 1, 2, \ldots \]  

Differentiating the left and right sides of (8) with respect to \( F_0 \), we find a system of ordinary differential equations:
\[ \dot{z}_n(Fo) = q(Fo), \]
\[ \left( \frac{1}{n \pi} \right)^2 z_n(Fo) + z_n(Fo) = 2 \left( \frac{1}{n \pi} \right)^2 q(Fo), \quad n = 1, 2, \ldots , \quad (9) \]

with the initial conditions
\[ z_o(0) = \int_0^1 f(X) dX, \]
\[ z_n(0) = (-1)^n 2 \int_0^1 f(X) \cos n \pi X dX, \quad n = 1, 2, \ldots \quad (10) \]

Substituting \( X = 1 \) into (7), and using \( \cos n \pi = (-1)^n \) \( (n = 1, 2, \ldots) \), we find
\[ \theta(1, Fo) = z_o(Fo) + \sum_{n=1}^{\infty} z_n(Fo). \quad (11) \]

Now, we use (5) and (11) to convert system (9) to
\[ \frac{1}{Bi(Fo)} \dot{z}_0(Fo) + z_0(Fo) = \theta_m(Fo) - \sum_{m=1}^{\infty} z_m(Fo), \]
\[ \left( \frac{1}{n \pi} \right)^2 z_n(Fo) + z_n(Fo) = 2 \frac{1}{Bi(Fo)} \left[ \theta_m(Fo) - z_0(Fo) - \sum_{m=1}^{\infty} z_m(Fo) \right], \quad n = 1, 2, \ldots \quad (12) \]

Solving this system with the initial conditions (10), we can determine from Eq. (7) the temperature at any point \( X \) of the plate.

For approximate calculations, we can retain a finite number of the differential equations in system (12), because the time constants \((1/n \pi)^2 \) \( (n = 1, 2, \ldots) \) decrease rapidly as \( n \) increases, and are negligible at large \( n \) \((1/ \pi^2 = 0.1013; 1/4 \pi^2 = 0.0253; 1/9 \pi^2 = 0.0112; 1/16 \pi^2 = 0.0063; 1/25 \pi^2 = 0.0040; 1/36 \pi^2 = 0.0028)\).

This truncated system of equations can be written
\[ \frac{1}{Bi(Fo)} \dot{z}_0(Fo) + z_0(Fo) = \theta_c(Fo) - \sum_{m=1}^{s} z_m(Fo), \]
\[ \left( \frac{1}{n \pi} \right)^2 z_n(Fo) + z_n(Fo) = 2 \frac{1}{Bi(Fo)} \left[ \theta_c(Fo) - z_0(Fo) - \sum_{m=1}^{s} z_m(Fo) \right], \quad n = 1, 2, \ldots , s, \quad (13) \]

From the solution of this system, we find the temperature at point \( X \) of the plate, using the equation
\[ \theta(X, Fo) = z_0(Fo) + \sum_{n=1}^{s} (-1)^n \cos n \pi X z_n(Fo) + 2 \sum_{n=s+1}^{\infty} (-1)^n \left( \frac{1}{n \pi} \right)^2 \cos n \pi X z_n(Fo). \quad (14) \]