BOUNDARY LAYER CALCULATION ON THE INSULATED WALLS OF A MHD GENERATOR CHANNEL WITH ALLOWANCE FOR HALL CURRENT

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An examination is made of the laminar boundary layer of low-temperature plasma on the insulated walls of a MHD generator channel with arbitrary magnetic field along the channel, perpendicular to the wall. An example of a boundary layer calculation by this method is given.

When an electrically conducting, low-temperature plasma flows in a MHD generator channel, the boundary layers formed near the walls and the electrodes have an appreciable influence on generator characteristics.

The present paper examines the laminar boundary layer on the insulated walls with allowance for Hall current. The method of calculation described enables some of the poorly founded assumptions used by Kerrebrock [1] to be avoided, particularly the assumption of local similarity.

1. Statement of the problem. A stream of low-temperature plasma moves in a rectangular channel with velocity \( u \) (Fig. 1). Two of the walls are insulated, the other two being electrodes. Motion occurs at right angles to the magnetic field, which has an arbitrary distribution along the channel. The electric current generated has two components: a current \( j_y \) directed across the channel, and a Hall current \( j_x \) along the channel axis. As a result of the Hall current, there is a transverse velocity component \( v \) in the boundary layer, causing appreciable three-dimensional effects in the boundary layer on the insulated walls.

The solution is carried out under the following assumptions:

1. The plasma stream is steady.
2. The flow is laminar.
3. The plasma is compressible.
4. \( R_e \ll 1 \). The magnetic field has one component along the \( z \) axis, and this is a function of \( x \).
5. The Hall coefficient is \( \beta = e / m_e \neq 0 \).

The equations describing the boundary layer may be simplified appreciably by taking into account the fact that the transverse velocity component \( v \) due to the Hall current is not large, and by omitting terms of order \( \beta \) and less in the equations, the boundary layer equations may be written as follows:

\[
\frac{\partial \psi}{\partial x} + \frac{\partial \varphi}{\partial z} = 0, \quad (1)
\]

\[
\frac{\partial \varphi}{\partial x} + \frac{\partial \psi}{\partial z} = -\frac{\partial p}{\partial x} + \frac{\partial}{\partial z} \left[ \mu \frac{\partial \varphi}{\partial z} \right] - j_x B, \quad (2)
\]

\[
\frac{\partial u}{\partial x} + \frac{\partial \psi}{\partial z} = \frac{\partial}{\partial z} \left[ \frac{\partial \varphi}{\partial z} \right] + j_x B, \quad (3)
\]

\[
\frac{\partial \varphi}{\partial x} + \frac{\partial \psi}{\partial z} = \frac{\partial^2 \psi}{\partial z^2} + \frac{\partial^2 \varphi}{\partial z^2} \left( \frac{u^2}{2} + \frac{v^2}{2} \right) + \frac{\partial}{\partial z} \left[ \mu \frac{\partial \psi}{\partial z} \right] - j_x E_y, \quad (4)
\]

with boundary conditions

\[
u = v = w = 0, \quad T = T_w \text{ when } z = 0,
\]

\[
u = u_x (x), \quad w = v = 0, \quad T = T_w \text{ when } z = \infty. \quad (8)
\]

The system of equations obtained coincides with that used in Kerrebrock's analysis [1]. We have managed to eliminate \( v \) from (3), and we may therefore examine the problem as a plane one, using the stream function \( \psi \).

From the solution of the problem, in the one-dimensional approximation, of the motion of an inviscid non-heat-conducting plasma in crossed electric and magnetic fields, under the condition \( T_w = \text{const} \), the flow parameters \( p(x), u_x(x), j_x(x) \) in the stream core are determined. We consider that there is no Hall current in the core, which is possible where there is an applied electric field along the flow axis

\[
E_x = \beta \frac{\partial x}{\partial x} (E_y - u_x B). \quad (9)
\]

We transform Eqs. (2)–(4) with the help of the stream core equations

\[
\frac{dp}{dx} = -j_x B - \rho \frac{du_x}{dx}, \quad (10)
\]
Substituting (10) and (11) into (2)-(4), and assuming \( n = 1 \) in (6), we obtain

\[
\rho u \frac{\partial u}{\partial x} + \rho w \frac{\partial u}{\partial z} = \rho u \frac{du}{dx} + \frac{\partial}{\partial z} \left( \rho \frac{du}{dx} \right) + B(j_x - j_y), \tag{2'}
\]

\[
\rho u \frac{\partial v}{\partial x} + \rho w \frac{\partial v}{\partial z} = \frac{\partial}{\partial z} \left( \rho \frac{dv}{dz} \right) + j_y B, \tag{3'}
\]

\[
\rho u \frac{\partial}{\partial x} \left[ C_p T + \frac{u_x^2 + u_z^2}{2} \right] + \rho w \frac{\partial}{\partial z} \left[ C_p T + \frac{u_x^2 + u_z^2}{2} \right] = \frac{\partial}{\partial z} \left[ \chi \frac{\partial T}{\partial z} \right] + \frac{\partial}{\partial z} \left[ \mu \frac{\partial u_x}{\partial z} \right] + j_x E_x - \rho u \rho_w \frac{du_x}{dx} \left. \frac{\partial}{\partial z} \frac{j_x}{j_\infty} \right|_{\eta} \tag{4'}
\]

2. Method of solution. The solution of (2')-(4') is effected by introducing into the equations the Dorodnitsyn variables \([1, 3]\)

\[
\bar{x} = \frac{x}{\xi}, \quad \eta = \frac{u_x}{2v_x \xi}, \quad \xi = \int_{\eta}^{\infty} \frac{\rho_x}{\rho} \, dz, \tag{12}
\]

Taking into account the experimental data obtained in flow of low-temperature plasmas in rectangular channels, and also the fact that the optimum construction for generators is a channel in the shape of a diffuser, the relation \( u_x(x) \) may be approximated with a high degree of accuracy by the "one-slope" velocity profile formula

\[
u_u(\bar{x}) = u_\infty(1 - \bar{x}^2) = u_\infty(1 - \bar{x}), \tag{13}
\]

where \( \xi < 1 \).

Let us introduce the stream function \( \psi \), satisfying the continuity equation (1)

\[
\rho \psi = \frac{\partial \psi}{\partial \zeta}, \quad \rho \omega = -\frac{\partial \psi}{\partial \xi}. \tag{14}
\]

Following Howarth* [2], we shall represent the stream function as a power series in \( \xi \):

\[
\psi = V \frac{2u_\infty v_\infty}{v_x} \int [f_0(\xi) - (8\xi^2) f_1(\xi) + (8\xi^2)^2 f_2(\xi) - \ldots] \tag{15}
\]

Similarly, we shall expand the dimensionless temperature \( \theta \) in a series of ascending powers of \( \xi \):

\[
\theta = T/T_\infty = \Theta_0(\xi) - (8\xi^2) \Theta_1(\xi) + (8\xi)^2 \Theta_2(\xi) - \ldots \tag{16}
\]

The stream velocities in the longitudinal and transverse directions may be written in the form

\[
\bar{u} = u/u_\infty = \frac{1}{2} \int [f_0(\xi) - (8\xi^2) f_1(\xi) + (8\xi)^2 f_2(\xi) - \ldots], \tag{17}**
\]

\[
\bar{v} = v/u_\infty = \int [- (8\xi^2) r_1(\xi) + (8\xi)^2 r_2(\xi) - \ldots]. \tag{18}
\]

Auxiliary equations [1] were used in the solution: for the electrical conductivity through the boundary layer

\[
\sigma = \sigma_\infty \exp \left[ - \chi (1/\Theta - 1) \right], \tag{19}
\]

(where \( \chi = \varphi / 2k \), \( \varphi \) is the ionization potential of the relevant alkali metal);

for the Hall coefficient under equilibrium conduction (19)

\[
\beta_x = \beta_\infty \Theta^{1/2}. \tag{20}
\]

*In calculating the boundary layers the Howarth method was used, first applied, as far as we know, to flow of compressible fluids in a magnetic field by Genkin [5].

**Here and below the prime denotes differentiation with respect to \( \eta \).