AVERAGING IN A SYSTEM WITH SEVERAL LIMIT CYCLES

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We consider the construction of the positive cone [7] by the method of averaging, which makes it possible to determine oscillatory modes in a many-frequency system with a polynomial nonlinearity and to construct curves dividing identical behavior of the trajectories. The starting point is the set of results given in [1-6].

1. Transformation of the Equations of Motion. We consider an autonomous system describing a certain oscillation process

\[ \dot{x} = Ax + \mu X(x), \quad (\cdot) = \frac{d}{dt}, \]  

where \( x(t) \in \mathbb{R}^{2n} \) for all \( t \in \mathbb{R} \), \( A \) is a \( 2n \times 2n \) constant matrix, \( 1 > \mu > 0 \) is a small parameter, \( X(x) : \mathbb{R}^{2n} \to \mathbb{R}^{2n} \) is a vector polynomial of integer powers first order and higher. Suppose that the eigenvalues of the matrix of the linear system corresponding to (1.1) are complex conjugates

\[ \lambda_j, \bar{\lambda}_j = \Re \lambda_j, \pm \Im \lambda_j \quad (j = \overline{1, n}). \]  

The conditions for which the variables of (1.1) can be separated into fast and slow variables are known (see [1], Sec. 163): \( |\Re \lambda_k| < < |\Im \lambda_k| \), \( k, k = 1, \ldots, n \). We write (1.1) in the form

\[ x = A_1 x + \mu [A_2 x + \Xi(x)]. \]  

where \( A_1 \) and \( A_2 \) are \( 2n \times 2n \) constant matrices and \( \Xi(x) \) is a vector polynomial containing terms second order and higher. The unperturbed form of (1.3) has the matrix \( A = A_1 \). Suppose that the roots of the matrix \( A_1 \) are imaginary and \( X(x) = A_2 + \Xi(x)\).

With the help of the nondegenerate linear transformation for (1.3)

\[ y_j = \sum_{k=1}^{2n} a_{jk} x_k, \quad \bar{y}_j = \sum_{k=1}^{2n} b_{jk} x_k, \quad x_k = \sum_{(j)} (a_{kj} y_j + \bar{a}_{kj} \bar{y}_j), \]

where \( a_{kj}, b_{jk}, \bar{a}_{kj}, \bar{b}_{jk} \) are constants, (1.1) is brought to diagonal form

\[ \dot{y}_j = \dot{\bar{y}}_j = \mu y_j + \bar{y}_j, \quad \bar{\dot{y}}_j = \bar{\dot{y}}_j = \mu \bar{y}_j + \mu \bar{y}_j, \quad (j = \overline{1, n}) \]

where \( Y_j = \sum_{(k)} \beta_{ik} X_k(y, \bar{y}), Y_j = \sum_{(k)} \bar{\beta}_{ik} X_k(y, \bar{y}); X_k(y, \bar{y}) \) is a vector function in which the vector \( x \) is expressed in terms of \( y \) and \( \bar{y} \).
In (1.4) we transform to the variables $\rho$, $\theta$ following [8], p. 96.

$$\rho_j = \gamma_j e^{-\tau_j}, \quad \rho_j = \gamma_j e^{i\theta_j}, \quad (j = 1, n).$$  \hspace{1cm} \text{(1.5)}$$

In terms of $\rho$, $\theta$ the equations of motion transform to

$$\dot{\rho} = \mu m(\theta) X(\rho, \theta), \quad \dot{\theta} = \text{Im} \Lambda \rho + \mu k(\theta) X(\rho, \theta).$$  \hspace{1cm} \text{(1.6)}$$

where $\text{Im} \Lambda$ is an $n \times n$ diagonal matrix and $m(\theta)$ and $k(\theta)$ are $n \times 2n$ matrices with elements

$$m_{j0} = \text{Re} \beta j_0 \cos \theta_j + \text{Im} \beta j_0 \sin \theta_j; \quad k_{j0} = \text{Im} \beta j_0 \cos \theta_j - \text{Re} \beta j_0 \sin \theta_j;$$

$X(\rho, \theta)$ is a vector function in which $X$ is expressed through $\rho$, $\theta$ as

$$x_k = 2 \sum_{o=1}^{n} \rho_o (\text{Re} \alpha_{k0} \cos \theta_o - \text{Im} \alpha_{k0} \sin \theta_o) \quad (k = 1, 2n).$$  \hspace{1cm} \text{(1.7)}$$

Note that in (1.6) the variables $\rho$ are identified with the vector of slow variables and $\theta$ with the vector of fast variables. The components of $\rho$ can be expressed in terms of $x$ and $\theta$ as

$$\rho_j = (\sum_{k=1}^{2n} \text{Re} \beta_{jk} x_k) \cos \theta_j + (\sum_{k=1}^{2n} \text{Im} \beta_{jk} x_k) \sin \theta_j \quad (j = 1, n).$$  \hspace{1cm} \text{(1.8)}$$

We represent (1.6) in the form

$$\dot{\rho} = \mu R(\rho, \theta), \quad \dot{\theta} = \text{Im} \lambda + \mu \Theta(\rho, \theta)$$  \hspace{1cm} \text{(1.9)}$$

and assume the initial conditions

$$\rho(0) = \langle \rho(0) \rangle = \rho_u, \quad \theta(0) = \langle \theta(0) \rangle = \theta_u.$$  

Here

$$R(\rho, \theta) = (R_1(\rho, \theta), ..., R_n(\rho, \theta)), \quad \Theta(\rho, \theta) = (\Theta_1(\rho, \theta), ..., \Theta_2(\rho, \theta)).$$

$$R_j(\rho, \theta) = \sum_{k=1}^{2n} m_{jk}(\theta) X_k(\rho, \theta), \quad \Theta_j(\rho, \theta) = \sum_{k=1}^{2n} k_{jk}(\theta) X_k(\rho, \theta) / \rho_j \quad (j = 1, n).$$  \hspace{1cm} \text{(1.10)}$$