APPLICATION OF PLANAR ELASTICITY EQUATIONS  
TO THE INVESTIGATION OF VIBRATIONS OF EXTENDED  
MULTILAYERED SLABS WITH INTERNAL LINE SUPPORTS  

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Previously proposed schemes for determining the stress—strain state of multilayered slabs with internal elastic supports [3, 5] are based on relations from the theory of small-curvature (almost-flat) shells. To determine the limits of validity of the results obtained in this case and to obtain a more reliable description of such mechanical systems, we need to recruit models without the simplifications of applied theories. This objective can be met by the dynamical elasticity equations used in several papers ([2, 6, 8-11] and others) to study vibrations of plates and shells. In the present article we augment the existing results by investigating the vibrations of an extended multilayered slab with internal supports (see Fig. 1). In contrast with the static elasticity problem treated earlier for a slab with regularly spaced beam supports [4], the application of standard approaches [8, 9] incurs patent difficulties in the attempt to satisfy the strain compatibility conditions for the slab and the supports. Consequently, our objective in this article is to develop efficient practical algorithms for determining the normal mode frequencies of the investigated mechanical system and to discuss the feasibility of using analogous schemes to eventually determine the mode shapes and characteristics of the stress—strain state for forced vibrations far from resonance points. This is accomplished by a separation-of-variables scheme with the subsequent analytical solution of a system of ordinary differential equations with constant coefficients, the use of so-called "transfer matrices" [8, 11] for multilayered media, and transformation of the matrix of an infinite system of linear algebraic equations. This procedure reduces the determination of the normal mode frequencies to the analysis of a determinant whose order depends only on the number of supports and not on the number of layers or the number of terms retained in the series representation of the solutions. The determination of the mode shapes and the case of forced vibrations far from resonance points require the additional solution of linear fourth-order systems and the summation of series.

1. We assume that a freely supported, extended, rectangular slab of width a and total thickness h, consisting of K layers and resting on an additional set of N internal elastic supports in the form of thin plates rigidly attached to a massive base, is subjected to planar deformation (see Fig. 1). The dynamical behavior of each layer can then be described by a system of differential equations of motion of an orthotropic body in displacements [6, 10]; if a harmonic time dependence is assumed in the equations, they are transformed to a system of vibration equations of the form

\[
\sum_{i=1}^{2} \left( \frac{\partial}{\partial z} + \rho \omega^2 \right) \frac{\partial u_i^{(j)}}{\partial t} = 0, \quad (j=1,2; \quad i=1,2,...,K),
\]

where \( L_{ij}^{(i)} \) denotes partial differential operators with constant coefficients, \( \rho^{(i)} \) is the density, \( \omega \) is the angular frequency, \( u_1^{(i)} \) and \( u_2^{(i)} \) are the components of the displacement vector, and the superscript refers to the i-th layer. The surface \( z = h \) is assumed to be unloaded (in the case of forced vibrations the load is assumed to be given). The surface \( z = z_0 \) is connected to the supports. Replacing the action of the supports by normal reactions \( P_j \) and tangential reactions \( Q_j \) (\( j = 1, 2,..., N \)) and using Dirac functions (in calculations the latter are conveniently replaced by sublimiting functions [1]), we can write the boundary conditions on the faces of the slab as follows:

\[
\overline{\sigma^{(i)}} \bigg|_{z=z_k} = 0.
\]
The following notation is used in Eqs. (2) and (3):

$$\bar{\sigma} = \sum_{j=1}^{N} \Delta(x - x_j) p_j$$

and $x_j$ denotes the coordinate of the $j$-th support. At the junction points of the slab with the supports the displacements of the surface of the slab and the ends of the supports $f_x^{(j)}$, $f_z^{(j)}$ must be equal, and they must first be expressed in terms of the unknown reactions:

$$u^{(i)}(x = x_j) = u^{(i+1)}(x = x_j) \quad i = 1, 2, \ldots, N$$

The layer contact conditions must be satisfied at the interfaces of the layers $z = z_i (i = 1, 2, \ldots, K - 1)$ [8, 9]:

$$u^{(i)}(x = x_j) = u^{(i+1)}(x = x_j) \quad i = 1, 2, \ldots, N - 1$$

Other specifications of the conditions of contact of the layers and slab with the supports are possible. For example, if the supports are hinged to the slab, it is required to set $Q_j = 0$ and $f_x^{(j)} = 0$ in Eqs. (3) and (4); if the layers are allowed to slip, we have $-u^{(i)}_1 = 0$ and $f_{z_{x}}^{(i)} = 0$ in Eqs. (5); for rigid supports we have $f_x^{(i)} = 0$ and $f_z^{(i)} = 0$.

A solution of the above-stated problem can be formulated by the separation of variables in cases where conditions corresponding to the Navier conditions in the theory of plates [2, 9] are satisfied at the edges of the slab $x = 0$ and $x = a$.

This means that the unknown displacements and stresses used in the formulation of conditions (2)-(5) can be represented by series ($\lambda_k = k\pi/a$ is the half-period of the expansions):

$$u^{(i)}(x) = \sum_k \left[ \cos \lambda_k x \begin{bmatrix} 0 \\ \sin \lambda_k x \end{bmatrix} e^{i \lambda k(x)} \right]$$

$$v^{(i)}(x) = \sum_k \left[ \sin \lambda_k x \begin{bmatrix} 0 \\ \cos \lambda_k x \end{bmatrix} e^{i \lambda k(x)} \right]$$

The substitution of the series (6) into the system (1) yields a family of independent systems of fourth-order ordinary differential equations with constant coefficients. The form of the solutions, which depend on four arbitrary constants $\gamma_{km}^{(i)}$ ($m = 1, \ldots, 4$), is determined by the roots of the characteristic equation $\xi_{km}^{(i)}$. For an isotropic layer the roots are always purely real or purely imaginary, but when anisotropy is present, the roots are complex-valued in general. The roots become exact multiples for the static problem ($\omega \to 0$). In the case of simple roots the displacements and stresses are written in the form

$$\left[ e^{(i)}_k(x) \right] = \sum_m \gamma_{km}^{(i)} \left[ 1 \right] \exp(\xi_{km}^{(i)} x)$$

where $\alpha_{km}^{(i)}$, $\beta_{km}^{(i)}$, and $\gamma_{km}^{(i)}$ are coefficients that depend on the elastic constants of the layer; $\xi_{km}^{(i)}$, $\alpha_{km}^{(i)}$, $\beta_{km}^{(i)}$, and $\gamma_{km}^{(i)}$ are functions of the frequency.

2. The principal stage in solving the problem is the satisfaction of conditions (2)-(5). The substitution of expressions (6)-(9) into these conditions yields an infinite system of algebraic equations in arbitrary constants and the reactions, where the coefficients of the system depend on the frequency. If $M$ harmonics are retained in the series (6) and (7), the total number of constants is $4MK + 2N$. They can be evaluated by using the contact conditions (5) to form $4M(K - 1)$ equations; once the delta functions have been expanded into trigonometric series, conditions (2) and (3) give another $4M$ equations, and the final $2N$ equations are obtained from conditions (4), which we can write as follows for rigid supports, representing the solutions in the form (6), (8):