ON BENDING WAVES LOCALIZED ALONG THE EDGE OF A PLATE

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We consider bending waves localized along the edge of a plate using nonclassical equations of motion and boundary conditions.

§1. A semi-infinite elastic plate of constant thickness occupies the region \(-\infty < x < \infty, 0 \leq y < \infty, -h \leq z \leq h\), where \(x, y, z\) are Cartesian coordinates. According to the theory of a thin plate based on the Kirchoff hypothesis and neglecting rotation, the transverse vibrations of the plate satisfy an equation of the form

\[
D \Delta^2 w + 2\rho h \frac{\partial^2 w}{\partial t^2} = 0, \quad D = \frac{2Eh^3}{3(1-\nu^2)},
\]

where \(w\) is the deflection, \(D\) is the stiffness, \(E\) is Young's modulus, \(\nu\) is the Poisson ratio, \(\rho\) is the density of the plate, and \(\Delta\) is the two-dimensional Laplacian.

We assume that the edge of the plate \(y = 0\) is free, i.e., bending moments and the generalized shear force are equal to zero. Then the boundary conditions take the form

\[
\frac{\partial w}{\partial y} + \nu \frac{\partial^2 w}{\partial x^2} = 0, \quad \frac{\partial^3 w}{\partial y^3} + (2-\nu) \frac{\partial^3 w}{\partial x^2 \partial y} = 0 \quad \text{at} \quad y = 0.
\]

It is required to find the solution of (1.1) which satisfies the boundary conditions (1.2) and the damping condition at infinity

\[
\lim_{y \to \infty} w = 0.
\]

The problem was first considered in this formulation by Konenkov [8], who showed the existence of a wave propagating in the \(x\) direction with an amplitude that decayed exponentially in the \(y\) direction. For brevity we will call this type of wave an edge wave, rather than a bending wave localized along the edge of the plate or a surface bending wave. Edge waves were also considered in [2-4, 7].

The general solution of (1.1) that satisfies (1.3) can be written in the form

\[
w = (c_1 e^{-k_1 y} + c_2 e^{-k_2 y}) \exp(i(\omega t - k x)),
\]

where

\[
\beta_1^2 = 1 + \eta, \quad \beta_2^2 = 1 - \eta, \quad \eta^2 = 2\rho h \omega^2 / (Dk^4), \quad k > 0
\]

and we must have

\[
0 < \eta^2 < 1.
\]

Substituting (1.6) into the boundary conditions (1.2) and requiring that the determinant of the resulting system of equations for the arbitrary constants \(c_1\) and \(c_2\) vanish, we obtain the dispersion relation

\[
K(\eta) = (\beta_1 - \beta_2)(\beta_1^2 \beta_2^2 + 2(1-\nu)\beta_1 \beta_2 - \nu^2) = 0.
\]
Noting that $\beta_1 = \beta_2$ only when $\eta = 0$ and that the root $\eta = 0$ of (1.7) corresponds to the trivial solution $W = 0$, we consider the following equation instead of (1.7):

$$K_1(\eta) = \beta_1^2 \beta_2^2 + 2(1 - \nu)\beta_1 - \nu^2 = 0.$$  

(1.8)

The function $K_1(\eta)$ has the following properties:

$$K_1(0) = (3 + \nu)(1 - \nu) > 0, \quad K_1(1) = -\nu^2 < 0, \quad K_1'(\eta) = 0.$$  

(1.9)

Because $-1 < \nu < 0.5$ in order for the elastic energy to be positive (the so-called natural values of $\nu$ are within the interval $0 < \nu < 0.5$), it follows from (1.9) that (1.8) subject to the condition $0 < \eta^2 < 1$ has a single real root if $\nu \neq 0$. Hence $\nu \neq 0$ is the condition for the existence of the edge wave.

In [3] the boundary conditions (1.2) were replaced with the conditions

$$\frac{\partial^2 w}{\partial y^2} + \nu \frac{\partial^2 w}{\partial x^2} = 0, \quad \frac{\partial}{\partial y} \Delta w = 0 \quad \text{at} \quad y = 0,$$  

(1.10)

which assume zero bending moment and zero shear force (rather than zero generalized shear force, as above) on the edge of the plate. In this case the dispersion relation becomes (after canceling out the factor $\beta_1 - \beta_2$)

$$K_1(\eta) = \beta_1 \beta_2 - \nu = 0,$$  

(1.11)

and hence the edge wave exists if $\nu > 0$, i.e., it always exists for natural values of the Poisson ratio.

If we assume zero bending and torsional moments on the edge of the plate:

$$\frac{\partial^2 w}{\partial y^2} + \nu \frac{\partial^2 w}{\partial x^2} = 0, \quad \frac{\partial^2 w}{\partial x \partial y} \Delta w = 0 \quad \text{at} \quad y = 0,$$  

then the dispersion relation will have the form

$$K_1(\eta) = \beta_1 \beta_2 + \nu = 0.$$  

(1.12)

Hence in this case the edge wave exists when $\nu < 0$, i.e., the edge wave can exist subject to the condition of positive elastic energy, but does not exist for natural values of $\nu$.

We note that edge waves do not exist if we assume a clamped edge or slip contact of the edge.

§2. It is of interest to consider edge waves using more exact theories that take into account shear deformations and rotational inertia. In addition, more realistic boundary conditions can be considered with the help of the more exact theories. In particular, we consider the case when the bending and torsional moments and the shear force all vanish on the free edge of the plate.

According to the theory of [1], the transverse shear stresses are

$$\sigma_{13} = f(z) \psi_1, \quad \sigma_{23} = f(z) \psi_2, \quad f(z) = 1 - z^2 / h^2$$  

(2.1)

and the equation of motion of a transversely isotropic plate can be written as

$$\Delta \Psi = \frac{3 \rho}{2} \frac{\partial^2 w}{\partial t^2};$$  

(2.2)

$$D \Delta^2 w - \frac{2 \rho h^3}{3} \left[ 1 + \frac{6E}{5(1-\nu^2)G} \right] \frac{\partial^2}{\partial t^2} \Delta w + \frac{4 \rho^2 h^3}{5G} \frac{\partial^4 w}{\partial t^4} + 2 \rho h \frac{\partial^2 w}{\partial t^2} = 0,$$  

(2.3)

where

$$\psi_1 = \frac{\partial \psi}{\partial x}, \quad \psi_2 = \frac{\partial \psi}{\partial y}.$$  

(2.4)

Assuming a solution of (2.3) in the form

$$w = W(y) \exp(\omega t - k x),$$  

(2.5)

we obtain the following equation for $W(y)$: