THE DISTRIBUTION FUNCTION OF ATOMIC LEVEL POPULATIONS IN A PLASMA

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An analytic form is obtained for the population distribution function in an atomic plasma as a series in successive time derivatives of the population of the first level. The first approximation includes the well-known method of a stationary sink. The quasistationary distribution obtained for hydrogen agrees well with numerical calculations of recombination and ionization, and for lithium, helium, and argon the quasistationary distribution gives qualitative agreement with numerical calculations.

The determination of the population distribution function of the discrete levels in an atomic plasma is of great importance in problems of ionization, recombination, and emission of plasmas and also the design and optimization of atomic-level lasers. Its role has been especially emphasized by the results of molecular kinetics, in which quasistationary distributions over the vibrational levels of molecules have recently been obtained. These are the Treanor distribution [1] and various generalizations thereof. The simple analytic form of such distributions had made it possible to establish a number of new laws in vibrational kinetics and has been widely used in laser theory [2-4].

In atomic kinetics the situation is more complicated. Here there are two approaches. One of them is based on a numerical solution of the balance equations for the populations. Very frequently, one has used a numerical solution of the simplified equations in the approximation of "stationary sink" (see, for example, [5-8]).

In the other approach [9-12], the motion of an electron between levels is treated as a stochastic process like Brownian motion in the energy space, the motion being described by the Fokker-Planck equation. In [11, 12], the authors used the modified diffusion approximation of the Fokker-Planck equation in finite differences and were able to preserve the real discrete structure of the energy spectrum. While the first approach is inconvenient because it is so cumbersome, the second, although entirely satisfactory for upper levels, is not accurate for the lower levels.

In the present paper we propose an analytic solution of the kinetic relaxation equations of a hydrogen plasma and, under certain simplifying assumptions, any atomic plasma. This solution is based on an expansion of the distribution in a series in the time derivatives of the population of the first level. In principle, any approximation can be calculated analytically, but in the majority of the cases of practical interest the first approximation is sufficient. The corresponding quasistationary distribution has, in some fundamental features, similarities with the Treanor distribution for molecules.

1. Method of Analytic Solution of the System of Kinetic Equations

Let us consider the bases on which we solve the equations of atomic kinetics, taking as an example the relaxation of a hydrogen plasma. We introduce simplifying assumptions. For the free electrons we assume a Maxwellian distribution function with constant electron temperature $T_e$. Of the various elementary collision processes, we take into account collisions of the first and the second kind of atoms with...
electrons and we ignore the infrequent processes of atom-atom and atom-ion collisions and also ionization, radiation, and triple recombination. In addition, the plasma is assumed to be optically thin and spatially homogeneous.

These assumptions enable us to write down the system of equations for the populations of the discrete levels of a hydrogen plasma without allowance for radiation decay of the levels as follows:

$$\frac{dN_n}{dt} = -V(n, n + 1)N_nN_e + V(n + 1, n)N_{n+1}N_e - V(n, n - 1)N_nN_e + V(n - 1, n)N_{n-1}N_e$$  \hspace{1cm} (1.1)

Here $N_n$ is the population of the $n$-th level of the hydrogen atom, $N_e$ is the electron density, and $V$ are the probabilities of collisions of the first and the second kind of electrons and atoms, averaged over the electron Maxwellian distribution. After times comparable with the collision time between free electrons, equilibrium with the continuous spectrum described by the Saha formula is established for the highest levels:

$$N_n = n^2N_e^2 \left( \frac{2\pi kT_e}{m_e} \right)^{3/2} \exp \left( \frac{E_{m+1} - mE_m}{kT_e} \right) \hspace{1cm} (n > n_0)$$  \hspace{1cm} (1.2)

where $R$ is the hydrogen ionization potential. The probabilities $V(m, m + 1)$ of the inverse processes are related to the probabilities of the direct processes by the principle of detailed balance:

$$\frac{V(m, m + 1)}{V(m + 1, m)} = \frac{(m + 1)^2}{m^2} \exp \left( \frac{E_{m+1} - mE_m}{kT_e} \right), \hspace{1cm} E_{m+1} = E_{m+1} - E_m.$$  \hspace{1cm} (1.3)

To solve the system (1.1) we make the change of variables

$$\tau = \int_0^t N_e(t') dt', \hspace{1cm} d\tau = N_e dt$$  \hspace{1cm} (1.4)

Summing the resulting equations from 1 to $m$, we obtain

$$N_{m+1} = \frac{V(m, m + 1)}{V(m + 1, m)} N_m + \frac{1}{V(m + 1, m)} \sum_{n=1}^m \frac{dN_n}{d\tau}$$  \hspace{1cm} (1.5)

The solution of the system (1.5) can be represented in the form

$$N_m = \sum_{i=0}^{m-1} \alpha_{m-1}^{i} N_1^{(i)}$$  \hspace{1cm} (1.6)

where the populations $N_m$ are determined by the derivatives of the populations of the first level $N_1^{(i)} = \frac{d^i}{dt^i} N_1$ and the quantities $\alpha_{m-1}^{i}$, which do not depend on $\tau$. To determine $\alpha_{m-1}^{i}$, we substitute (1.6) into (1.5):

$$\sum_{i=0}^m \alpha_{m-1}^{i} N_1^{(i)} = \frac{V(m, m + 1)}{V(m + 1, m)} \sum_{i=0}^m \alpha_{m}^{i} N_1^{(i)} + \frac{1}{V(m + 1, m)} \sum_{n=1}^m \sum_{i=1}^n \alpha_{n-1}^{i} N_1^{(i)}$$  \hspace{1cm} (1.7)

Reversing the order of summation in the second term in accordance with the formula

$$\sum_{n=1}^m \sum_{i=1}^n \rightarrow \sum_{i=1}^m \sum_{n=i}^m$$

and equating coefficients of corresponding derivatives, we obtain the recursion relations

$$\alpha_{m}^{i} = \frac{V(m, m + 1)}{V(m + 1, m)} \alpha_{m-1}^{i} + \frac{1}{V(m + 1, m)} \sum_{n=1}^m \alpha_{n-1}^{i-1}$$  \hspace{1cm} (1.8)

which can also be represented as

$$\alpha_{n}^{i} = \sum_{m=1}^n \frac{n^2 \exp (-E_n, m \alpha_T)}{V(m, m - 1)} \sum_{k=i}^{m-1} \alpha_{k-1}^{i}$$  \hspace{1cm} (1.9)