STABILITY OF A PLANE-PARALLEL CONVECTIVE FLOW OF A LIQUID IN A HORIZONTAL LAYER

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We consider the stationary plane-parallel convective flow, studied in [1], which appears in a two-dimensional horizontal layer of a liquid in the presence of a longitudinal temperature gradient. In the present paper we examine the stability of this flow relative to small perturbations. To solve the spectral amplitude problem and to determine the stability boundaries we apply a version of the Galerkin method, which was used earlier for studying the stability of convective flows in vertical and inclined layers in the presence of a transverse temperature difference or of internal heat sources (see [2]). A horizontal plane-parallel flow is found to be unstable relative to two critical modes of perturbations. For small Prandtl numbers the instability has a hydrodynamic character and is associated with the development of vortices on the boundary of counterflows. For moderate and for large Prandtl numbers the instability has a Rayleigh character and is due to a thermal stratification arising in the stationary flow.

1. Stationary Flow. We consider a two-dimensional horizontal layer of a liquid, bounded by the solid planes $x = \pm h$. On the two planes the temperature is given and varies linearly with the coordinate $z$:

$$\tag{1.1} T_0 = Ax$$

In a sufficiently long layer a plane-parallel stationary flow appears, having the following structure:

$$v_x = v_y = 0, \quad v_z = v_0(z), \quad T_0 = Ax + \tau_0(z), \quad p = p_0(x, z)$$

The stationary velocity, temperature, and pressure distributions may be obtained from the equations

$$\frac{1}{\rho} \frac{\partial \rho u}{\partial x} = g\beta (Az + \tau_0), \quad \frac{1}{\rho} \frac{\partial \rho T}{\partial z} = \nu \frac{\partial^2 u}{\partial z^2}, \quad \frac{\partial^2 u}{\partial z^2} = \chi T_0$$

$$\tag{1.3}$$

Here $\rho$ is the average density, $g$ is the gravitational acceleration, and $\nu$, $\chi$, and $\beta$ are, respectively, the coefficients of kinematic viscosity, thermal diffusivity, and thermal expansion. On the boundaries of the layer we have

$$v_0 = 0, \quad \tau_0 = 0 \quad (x = \pm h)$$

$$\tag{1.4}$$

In addition, we assume the closed flow condition to be satisfied:

$$\int_{-h}^{h} v_0 dx = 0$$

$$\tag{1.5}$$

From the Eqs. (1.3), with the boundary conditions (1.4) and (1.5), we determine the stationary flow [1]

$$v_0 = \frac{g3.4h}{6\nu} \left[ \frac{x}{h} \right] \left[ \frac{x}{h} \right] - \left[ \frac{x}{h} \right]$$

$$\tag{1.6}$$
The flow consists of two horizontal counter flows. The form of the velocity distribution turns out to be of the kind which appears in the case of a vertical layer, the boundaries of which are maintained at different temperatures. It can be expected that for a sufficiently large pressure intensity (i.e., for a sufficiently large gradient \( A \)) a hydrodynamic type of instability will arise.

From the distribution of temperature it follows that, although for a given \( z \) there is no transverse temperature difference between the layer boundaries, the flow leads to the formation of two layers in the liquid, in the interior of which there is a potentially unstable temperature stratification. These layers are located close to the upper and lower boundary planes. For a sufficiently large vertical temperature difference in these layers (proportional to \( A^2 \)) we can expect an instability of Rayleigh type to appear.

2. Equations for the Perturbations. Method of Solution. To study the stability of the stationary regime (1.6), we consider the perturbed flow \( v_0 + v, T_0 + T, p_0 + p \), where \( v, T, \) and \( p \) are small perturbations of the plane-parallel flow. We introduce the following units of measurement for distance, time, velocity, temperature, and pressure: \( h, h^2/\nu, g \beta h^4/\nu^2, Ah \) and \( \rho g \beta Ah^2 \). In dimensionless form the equations for small perturbations are

\[
\begin{align*}
\frac{\partial v}{\partial t} + G (\nu v) v_0 + (v_0 v) v &= - \nabla p + \Delta v + T \gamma \\
\frac{\partial T}{\partial t} + G (v v T) v_0 + (v_0 v) T &= P^{-1} \Delta T \\
\text{div } v &= 0
\end{align*}
\]

Here \( \gamma \) is a unit vector, directed vertically upwards, \( G = g \beta h^4/\nu^2 \) is the Grashof number, \( P = \nu / \chi \) is the Prandtl number, and \( v_0 \) and \( T_0 \) are the dimensionless unperturbed velocity and temperature profiles

\[
v_0 = 1/6 (x^3 - x), \quad T_0 = z + GP \tau_0, \quad \tau_0 = 1/360 (3x^4 - 10x^2 + 7x)
\]

The perturbations satisfy the homogeneous boundary conditions

\[
v = 0, \quad T = 0 \quad (x = \pm 1)
\]

We consider two-dimensional perturbations. In this case the velocity components \( v_x \) and \( v_z \) are different from zero and all the quantities are independent of the coordinate \( y \). Introducing "normal" perturbations of the stream function and of the temperature, namely,

\[
\psi = \varphi (x) \exp (-\lambda t + ikz), \quad T = \theta (x) \exp (-\lambda t + ikz)
\]

we obtain for the amplitudes \( \varphi \) and \( \theta \) the spectral problem

\[
\begin{align*}
\Delta \varphi - ikG (v_0 \Delta \varphi - v_0^* \varphi) - ik\theta &= - \lambda \Delta \varphi (\Delta \equiv d^2 / dx^2 - k^2) \\
P^{-1} \Delta \theta - ikG (v_0 \theta - G \psi \theta) - G \varphi &= - \lambda \theta \\
\varphi &= \varphi^* = 0, \quad \theta = 0 \quad (x = \pm 1)
\end{align*}
\]

The decrement \( \lambda \), depending on the parameters \( G, P, \) and the wave number \( k \), is a characteristic number of the boundary-value problem (2.7)-(2.9).

We note here the differences between the boundary-value problem (2.7)-(2.9) and the problem which arises in studying the stability of a stationary convective flow between vertical planes, heated to a different temperature [2]. One of these differences is that in the heat conduction Eq. (2.8) a new term, \( G \varphi' \), is present; this term describes the convective heat transfer in the field of the longitudinally unperturbed temperature gradient. The second difference is in the form of the lift force term in the equation of motion (2.7); in the case considered the lift force is perpendicular to the planes. Another difference is that the unperturbed temperature has a more involved distribution over a cross-section.

Just as in the case of a flow between parallel plates heated to different temperatures [3-6], we solve the amplitude problem by applying Galerkin's method. We write the amplitudes \( \varphi \) and \( \theta \) in the form of the expansions