PROPAGATION OF AN ELECTROMAGNETIC WAVE ACROSS A MAGNETIC FIELD IN A PARABOLIC PLASMA LAYER

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The reflection and transmission coefficients are obtained, as well as the coefficient of transformation of an electromagnetic wave into a plasma wave. The problem of choosing the "physical" path of analytic continuation of the solutions is considered in the case of a wave equation with two poles.

1. Statement of the Problem. Let a plane wave be propagated along the z axis, while the plasma is likewise inhomogeneous along the z axis, and the external magnetic field is directed along the y axis. In this case, the electric field of the wave

\[ E_x(z, t) = E_x(s)e^{iat} \]

can be described by the equation [1]

\[ \frac{d^2E_x}{dz^2} + \omega_0^2 \left[ 1 - \frac{v(1 - is - v)}{(1 - is)^2 - u(1 - is)\nu} \right] E_x = 0, \quad i = \sqrt{-1} \]  
\[ \nu = g(1 - z^2/z_m^2), \quad g = \omega_0^2/\omega, \quad u = \omega_H^2/\omega^2, \quad s = \nu_{ef} / \omega \]  

Here \( \omega \) is the cyclic frequency; \( \omega_H \) is the gyrofrequency for electrons; \( \omega_k \) is the plasma frequency at the maximum of the layer; \( \nu_{ef} \) is the effective number of collisions; \( z_m \) is the half-thickness of the layer; \( c \) is the velocity of light. Below we shall assume that \( u \) and \( s \) do not depend on \( z \).

The propagation of the ordinary and extraordinary waves in a parabolic layer was considered in the geometric-optics approximation in [2]. However, in [2], the effect of regions where geometric optics is violated was not considered. The principal content of the subsequent analysis is precisely the consideration of the effect of poles in the coefficient of \( E_x \) on the propagation of the wave. The case of a linear layer was considered in [3, 4].

2. Asymptotic Solutions. Let us introduce the new independent variable \( \tau = z/z_m \) in equations (1.1). Then this system of equations is written as

\[ (\tau^2 - \tau_1^2) \frac{d^2E_x}{d\tau^2} + \left[ p(\tau^2 - 1)^2 + q(2\tau^2 - \tau_1^2 - 1) \right] E_x = 0 \]  
\[ \tau_1^2 = 1 + \frac{u(1 - is)^2}{g(1 - is)}, \quad q = \left( \frac{\nu_{ef} \omega}{c} \right)^2, \quad p = \frac{qe^2}{1 - is} \]

The two regular singular points of the equation \( \tau = \pm \tau_1 \) merge into each other for \( \tau_1 = 0 \) (\( s = 0 \), \( \omega^2 = \omega_H^2 + \omega_k^2 \)). For \( \tau_1^2 = 1 \) (\( s = 0 \), \( \omega^2 = \omega_H^2 \)) the regular singular points are absent, which is quite understandable physically, since at the points \( \tau = \pm 1 \) the plasma density is equal to zero. As far as the value \( \omega = \omega_H \) is concerned, this value of \( \omega \) is not isolated in Eq. (1.1), which is associated with the approximation in which the equation was derived.

As is well known [5], solutions \( E_x^{(1)}, E_x^{(2)} \) of Eq. (2.1) exist which have the following asymptotic representations for fulfillment of the conditions \( |\tau| > |\tau_1|, |\sqrt{p\tau^2}| > 1 \):

The condition \(|\tau| > |\tau_1|\) leads to the condition \(|\tau_1| < 1\), which restricts the range of frequencies considered. In particular, for \(s = 0\), the allowable frequency range is

\[\omega_H^2 \leq \omega^2 < \omega_H^2 + 2\omega_k^2\]

If the incident wave propagates from the direction \(\tau < 0\), then \(E_x^{(2)}(\tau > 0)\) describes the transmitted wave. Correspondingly, for \(\tau < 0\) the quantity \(E_x^{(2)}\) conversely represents the reflected wave and \(E_x^{(1)}\) represents the incident wave.

In order to determine the amplitude coefficients of reflection (R) and transmission (D) it is necessary to know the relationship between the asymptotic solutions \(E_x^{(2)}\) for \(\tau > 0\), and \(E_x^{(1)}, E_x^{(2)}\) for \(\tau < 0\) (\(|\tau| > |\tau_1|\)). This relationship was established in [6] for a certain equation of which (2.1) is a particular case. Making use of the results of [6], one may write

\[
D = \frac{1}{2} + \frac{\tau_1^2 - p^2}{4\beta} F + e^{-\beta \tau_1} D (2.2)
\]

\[
R = e^{\beta \tau_1} (q_{+2} \cos 2\pi \tau_1 + e^{-\beta \tau_1}) D
\]

\[
\eta = \frac{1}{4} (\tau_1 - \tau), \quad q_+ = 1, \quad q_- = 0
\]

Here \(\Gamma\) is a gamma function; \(q_+\) corresponds to bypassing of the singular points \(\tau = \pm \tau_1\) (for transition from \(\tau > 0\) to \(\tau < 0\)) along the upper half plane of the complex \(\tau\) plane; \(q_-\) corresponds to bypassing of the singular points along the lower half plane; \(\mu\) is determined by the character of the singular points \(\tau = \pm \tau_1\).

3. Determination of \(\mu\). The solutions \(y_1, y_2\) of Eq. (2.1) have the following form [5] in the neighborhood of the angular points \(\tau = \pm \tau_1\):

\[
y_1 = (\tau + \tau_1) \sum_{n=0}^{\infty} C_n (\tau + \tau_1)^n
\]

\[
y_2 = \mp b y_1 \ln (\tau + \tau_1) + \sum_{n=0}^{\infty} d_n (\tau + \tau_1)^n
\]

\[
C_0 = 1, \quad d_0 = -1, \quad b = - (\tau_1^2 - 1) [p (\tau_1^2 - 1) + q] / 2\tau_1
\]

\[
C_{-1} (\tau_1) = - \frac{1}{2v (v - 1) \tau_1} \sum_{v=0}^{2\tau_2 - 2} C_v g_n (v), \quad k_0 = \begin{cases} 0 & (v \leq 0), \\ v - 6 & (v > 0) \end{cases} (v = 2, 3, \ldots)
\]

\[
g_{+2} (v) = (v - 1) (v - 2) + (\tau_1^2 - 1) [p (\tau_1^2 - 1) + q], \quad g_{+3} = 4pr_1
\]

\[
g_{+4} (v) = 4r_1 [p (\tau_1^2 - 1) + q], \quad g_{-4} (v) = 2 [p (3\tau_1^2 - 1) + q]
\]

Here \(b\) is determined by the Frobenius method [5]. As is easily demonstrated from recurrent relationships, the coefficient \(C_{-1}\) is an even function of \(\tau_1\), \(\tau_1\) is even, and an odd function if \(\nu\) is odd. As is evident from (3.1), the field \(E_x\) at the actual point of the pole \((\tau_1 = 0)\) is finite. Making use of the procedure developed in [6], we have

\[
\mu' = \frac{1}{8\pi^2} \ln \delta + \frac{1}{4}, \quad \delta = \frac{2 - \beta + \sqrt{\beta (4 - \beta)}}{2 - \beta - \sqrt{\beta (4 - \beta)}}
\]

\[
\beta = \left\{ 4\pi b \left[ y_1 \frac{dy_1}{d\tau} \right]_{\tau=0}^2 \right\}^{1/2}
\]

Here \(l\) is an integer which is to be determined. If the equation has only one regular singular point (or none at all), then \(\mu' = \frac{1}{4} (\theta_1 - \rho_2)\) in accordance with [6], where \(\rho_1, \rho_2\) is the solution of the defining equations for the case considered. For Eq. (2.1) we have

\[
\mu'_{(\tau_1 = 0)} = \frac{1}{4} \sqrt{1 + 4\pi^2 \omega^2 c^2}, \quad \mu'_{(\tau_1 = 1)} = \frac{1}{4}
\]