CALCULATION OF THE PRINCIPAL PARAMETERS
OF FREE SUPERSONIC JETS OF AN IDEAL
COMPRESSIBLE FLUID

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An approximate method of determining the principal parameters of axially symmetrical under-
expanded jets is set out. The results are presented in the form of computing formulas for
the geometrical characteristics of the configuration of the shock waves and the boundaries
of the jet. On the basis of experience with these calculations, some simple approximating
relationships are recommended.

Notation: x, y, rectangular coordinates in the plane of the axial section; p, pressure; p, density; w,
modulus of the velocity; M, Mach number; k, polytropy index; n, degree of wastage of the jet; α, Mach
angle; ϕ, angle between the velocity vector and the symmetry axis; ϕa, aperture semiangle of the nozzle in
the outlet section; θ, angle of rotation of the flow in the shock wave; ϕ, inclination of the leading edge of
the shock wave to the symmetry axis; ω, angle between the incident jump and the velocity vector of the in-
cident flow; K, curvature of the current lines; R, radius of curvature.

The indices denote: a, parameters in the outlet section of the nozzle; *, parameters at the boundary
of the jet; 1, parameters in front of the shock-wave branch point; 1, parameters at the vertex of the angle
between the leading edges of the branched jumps; 2, parameters behind the central jump. The dimension-
less gas-dynamic quantities are referred to the corresponding retardation parameters at the outlet from
the nozzle, the quantities with dimensions of velocity to the maximum velocity of steady-state outflow into
a vacuum. The linear quantities are expressed in terms of the radius of the outlet cross section of the
nozzle.

1. Numerical calculations of the shape of the boundaries in supersonic jets of a nonviscous, nonheat-
conducting gas are laborious and cumbersome. Simple approximate methods of solving this problem are
therefore very much to be sought. One such method was proposed in [1]; it was based on expanding the
solution in series with respect to even powers of the angle ϕ. The expansion is achieved by transforming
to an auxiliary plane of independent variables (pressure-current function) [2]. The initial approximation
then gives a unidimensional solution for flows in a channel with a variable cross-sectional area. In the
next approximation, the isobars (including the isobaric current line) form a family of second-order curves.
However, since the flow closely resembles a plane flow around an obstacle in the neighborhood of the sharp
edge [3] (up to distances of the order of the radius of the nozzle from the center of expansion), the generator
of the jet boundary differs little from a straight line up to distances of the same order. We may therefore
employ a twofold analytical specification for the shape of the boundary generator in different regions

\[ y = \begin{cases} 
1 + (t g \theta^2)x, & x \leq x^0 \\
\sqrt{A + Bx + Cx^2}, & x \geq x^0 
\end{cases} \]  

(1.1)

where A, B, C, x^0, \theta^0 are parameters reflecting the influence of the specific conditions of outflow and the
internal structure of the jet. The parameters A, B, C, and x^0 were defined in [1] in terms of the coordinates
x^*, y^* of the branch point of the shock waves and the pressure p^ at the tip of the angle between the incident
and reflected jumps. The following conditions and assumptions also apply:
a) smooth joining of the sections of the jet boundary generator is assumed at the point \((x^*, y^*)\)

\[
y^* = 1 + x^* \tan \theta^* = \sqrt{A + Bx^* + Cx^2} \\
\frac{dy^*}{dx} = \tan \theta^* = \frac{B + 2Cx^*}{2y^*}
\]

b) the active cross section of the flow passing through the contour of the central jump is regarded as approximately plane and normal to the symmetry axis; since this section lies fairly close to the maximal cross section, the direction of the velocity vector in the former differs little from axial, and the angle \(\phi\) is therefore regarded as small within this section;

c) allowing for the foregoing approximation, an integral form is taken for the law of conservation governing the mass of the gas;

d) it is considered that the curvature of the current lines \(K\) and the gradients of the pressure \(p\) remain finite within the region of flow considered and constitute reasonably smooth functions of distance from the boundary of the jet;

e) in the calculations, these functions are approximated by a quadratic relationship for \(p\) and a linear relationship for \(K\).

In this way we obtain an equation in \(x^*\) and formulas for the coefficients \(A, B,\) and \(C\)

\[
y_m^2 - 2(x_m - x^*)(1 + x^* \tan \theta^*) \tan \theta^* - (x_m - x^*)^2 [\tan^2 \theta^* + \frac{2y_m^3(p_m/p^* - 1)}{kM^2(1 + x^* \tan \theta^*)^2(y_m - y^*)}] = 0
\]

\[
A = y_m^3 - Bx_m - Cx_m^2 = (1 + x^* \tan \theta^*)^2 - Bx^* - Cx^2
\]

\[
B = 2[(1 + x^* \tan \theta^*) \tan \theta^* - Cx^2]
\]

\[
C = (x_m - x^*)^2[y_m^2 - 2(x_m - x^*)(1 + x^* \tan \theta^*) \tan \theta^* - (1 + x^* \tan \theta^*)^2]
\]

where

\[
y_m = -\frac{y_m}{2k} \left[ \frac{1}{k} \left( \frac{p_m}{p^*} - 1 \right) + 2 \right]^{-1} \left( \frac{p_m}{p^*} - 1 \right) - \frac{kM^2}{y_m} \left[ \frac{y_m^3}{2kM^2} \left( \frac{p_m}{p^*} - 1 \right)^2 + \frac{3}{M^2} \left( \frac{p_m}{p^*} - 1 \right) + 2 \right]
\]

\[
+ \frac{6D}{p^*} \left( \frac{2}{M^2} + k - 1 \right) \right]
\]

\[
D = \frac{1}{2} \left[ p_m \left( \sqrt{1 - p_m^{-1}} - 1 \right) + \frac{k + 1}{2k} p_m - \left( \sqrt{1 - p_m^{-1}} - 1 \right) \frac{k + 1}{y_m^2} p_m + \frac{k - 1}{2k} (1 - y_m^2) p^* \right] \quad \left( \gamma = \frac{4}{3} \right)
\]

Figure 1 illustrates various comparisons between the calculation of the boundary generator based on Eqs. (1.1) and (1.2) (broken lines) and the calculation based on the method of characteristics [3] (continuous line). In this we took \(k = 1.4, M_1 = 1.5, \phi = 0\).

Analogous results were obtained for other values of the original parameters \((1 \leq M_1 \leq 3.5, 5 \leq n \leq 25)\).

2. In an underexpanded jet, the first shock wave, or the suspended jump (curve 2 in Fig. 2), is generated at a certain distance from the nozzle cutoff, at the point \(O_1\), at which the characteristics of the second family of the second family of the jet 1 first intersect. The coordinates of this point may easily be expressed in terms of the radius of curvature \(R^*\) of the initial element of the jet boundary on the assumption that the characteristics are practically linear up to the intersection point.

A simple approximate formula may be obtained for \(R^*\) from the equation of motion projected on the normal to the current line, allowing for the continuity equation...