TURBULENT WAKE IN A STRATIFIED MEDIUM

A. T. Onufriev

The problem of determining the form of the turbulent wake being formed behind a self-propelled body in a medium with density varying in the direction of the effect of gravity is considered. The schematic picture of wake development behind a moving object is the following: Initially, diffusion is identical in all directions, and the wake broadens symmetrically; diffusion becomes strongly anisotropic with recession from the object, it diminishes in the vertical direction under the effect of gravity, and the wake becomes flattened; turbulent mixing within the wake results in the production of a more homogeneous density distribution within the volume occupied by the wake than in the surrounding medium; such a fluid volume turns out to be removed from the equilibrium state and tends to return to the equilibrium state under the effect of gravity; collapse of the wake occurs accompanied by its further expansion in the horizontal direction and the excitation of internal waves.

The problem of the first stage of wake development (prior to collapse), i.e., the problem of turbulent diffusion in a stratified medium, is considered herein. The medium itself is at rest. Equations obtained in [1], which permit the description of anisotropic diffusion, are the basis for the description of the diffusion process. The standard simplifications used in problems involving the propagation of a turbulent wake (reflecting the experimental observation that the free turbulent zones are relatively narrow) are used. Molecular diffusion is not taken into account. It is considered that density pulsations are small and, hence, should be taken into account only in members containing the acceleration of gravity [2].

The z axis is directed upward along the direction of the effect of gravity, and the x axis in the motion direction. The picture of the flow in the coordinate system connected with the body is stationary. The free stream velocity \( U_0 \) is assumed to be considerably greater than the velocity component in the wake. The scales along the axes \( l_x, l_y, l_z \) and the characteristic values of the pulsating velocities are distinct. The above-mentioned circumstances permit introduction of simplifications which are utilized in deriving the boundary layer equations. Moreover, the following simplifications are introduced: a) third-order moments corresponding to diffusion processes are neglected in all the equations; this is the customary assumption in the wake problem, otherwise it is necessary to make an assumption on the relation between the third-order moments and the average stream characteristics; b) constancy of the scale of turbulence \( L \) and the pulsating motion energy \( E \) in the wake cross section is assumed so that they vary only as functions of the longitudinal coordinate.

The change in \( E \) across the wake can be considered additionally.

The system of equations is

\[
\begin{align*}
U_0 \frac{\partial U}{\partial x} &= -\frac{\partial}{\partial y} \langle u_x u_y' \rangle - \frac{\partial}{\partial z} \langle u_x u_z' \rangle \\
\langle u_x u_y' \rangle &= -\frac{1}{2} \left[ \langle u_x'^2 \rangle \frac{\partial U}{\partial y} + \langle u_x' u_y' \rangle \frac{\partial U}{\partial x} + U_0 \frac{\partial}{\partial z} \langle u_x u_y' \rangle \right] \\
\langle u_x u_z' \rangle &= -\frac{1}{2} \left[ g \frac{\partial U}{\partial z} + \langle u_x' u_z' \rangle \frac{\partial U}{\partial y} + \right. \\
&\quad \left. + \langle u_z' u_z' \rangle \frac{\partial U}{\partial x} + U_0 \frac{\partial}{\partial z} \langle u_x u_z' \rangle \right]
\end{align*}
\]
The quantities \{\} are averaged over the wake cross section; \(A = 3.86, \alpha_L = 0.8\) [1], \(\alpha\) are empirical constants.

As is done in the case of wake diffusion in a homogeneous medium [3, 4], the solution is sought in the form of power-series dependences on the longitudinal coordinate.

Estimates (members containing the derivatives with respect to \(x\) with the factor \(U_0\) are not small quantities and, since \(\tau \sim x\), taking them into account results in small corrections) can be obtained on the basis of the system of equations

\[
\langle u_y'^2 \rangle = \frac{\tau}{2} \left[ g \frac{\langle u_y'^2 \rangle}{\rho} + U_0 \frac{\partial}{\partial z} \langle u_y'^2 \rangle \right]
\]

\[
\langle u_x'^2 \rangle = \frac{\tau}{3} \left[ \frac{\partial}{\partial y} \langle u_x' u_x' \rangle + \frac{\partial}{\partial z} \langle u_y' u_x' \rangle \right]
\]

\[
\langle u_y'^2 \rangle = \frac{2}{3} \left[ \frac{\partial}{\partial z} \langle u_x'^2 \rangle + U_0 \frac{\partial}{\partial z} \langle u_y'^2 \rangle \right]
\]

\[
\langle u_x'^2 \rangle = \frac{2}{3} \left[ \frac{\partial}{\partial z} \langle u_x'^2 \rangle + U_0 \frac{\partial}{\partial z} \langle u_y'^2 \rangle \right]
\]

\[
\langle u_y'^2 \rangle = \frac{E}{3} - \frac{\tau}{U_0 \frac{\partial}{\partial z} \langle u_y'^2 \rangle}
\]

\[
\langle u_x'^2 \rangle = \frac{E}{3} - \frac{\tau}{U_0 \frac{\partial}{\partial z} \langle u_x'^2 \rangle}
\]

\[
U_0 \frac{\partial E}{\partial z} + \left[ \langle u_x'^2 \rangle \frac{\partial}{\partial y} + \langle u_y'^2 \rangle \frac{\partial}{\partial z} \right] = - \frac{E}{\tau} - \frac{\tau}{U_0 \frac{\partial}{\partial z} \langle u_y'^2 \rangle}
\]

\[
\left[ \langle u_x'^2 \rangle \frac{\partial}{\partial y} + \langle u_y'^2 \rangle \frac{\partial}{\partial z} \right] + U_0 \frac{\partial}{\partial z} L E = - \frac{E}{\tau} - \frac{\tau}{U_0 \frac{\partial}{\partial z} \langle u_y'^2 \rangle}
\]

\[
\tau = A L E^{-\alpha_L}
\]

The quantity \(\langle u_y'^2 \rangle \approx - \frac{E}{6} \frac{\tau}{U_0 \frac{\partial}{\partial z} \langle u_y'^2 \rangle} \) \(N^2\), i.e., in the approximation under consideration, is

\[
\langle u_y'^2 \rangle = 0, \quad \langle u_x'^2 \rangle = - \frac{\partial}{\partial y} \psi, \quad \langle u_y'^2 \rangle = - \frac{\partial}{\partial z} \psi (\tau N) = \frac{1 - \frac{\tau}{U_0 \frac{\partial}{\partial z} \langle u_y'^2 \rangle}}{(1 + \frac{\tau}{U_0 \frac{\partial}{\partial z} \langle u_y'^2 \rangle})^2}
\]

The following equations are obtained for \(U, E, L\)

\[
U_0 \frac{\partial U}{\partial z} = \frac{\partial}{\partial y} \left[ \frac{\partial}{\partial y} \left( \frac{\partial}{\partial z} \right) \right] + \frac{\partial}{\partial z} \left[ \frac{\partial}{\partial z} \left( \frac{\partial}{\partial z} \right) \right]
\]

\[
U_0 \frac{\partial E}{\partial z} = \left[ \frac{\partial}{\partial y} \left( \frac{\partial}{\partial z} \right) + \frac{\partial}{\partial z} \left( \frac{\partial}{\partial z} \right) \right] = - \frac{E}{\tau} - \frac{\tau}{U_0 \frac{\partial}{\partial z} \langle u_y'^2 \rangle}
\]

\[
U_0 \frac{\partial L E}{\partial z} = \frac{\partial}{\partial z} \left[ \frac{\partial}{\partial y} \left( \frac{\partial}{\partial z} \right) - \frac{\partial}{\partial z} \left( \frac{\partial}{\partial z} \right) \right] = - \frac{E L}{\tau} - \frac{\tau}{U_0 \frac{\partial}{\partial z} \langle u_y'^2 \rangle}
\]

\[
\tau = A L E^{-\alpha_L}
\]

The approximate condition [4]

\[
\int U y^2 dy dx = \text{const}, \quad \int U z^2 dy dz = \text{const}
\]

must be added to these equations, which yields in the axisymmetric case

\[
\int U r^2 dr = \text{const}
\]