DETERMINATION OF UNSTEADY-STATE TEMPERATURE FIELDS IN MULTILAYERED ORTHOTOPIC PLATES

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A new hypothesis on temperature distribution with respect to the thickness of a multilayered plate is proposed. An analytical solution of the unsteady-state heat conduction problem is obtained for rectangular multilayered orthotropic plates.

To determine temperature fields in multilayered structures, different hypotheses on temperature distribution with respect to a multilayered plate are widely used. Such an approach makes it possible to reduce an initial three-dimensional problem to a two-dimensional one. Most often the hypothesis on a piecewise-linear temperature distribution is employed [2]. For a more accurate description of a temperature field, the polynomial law is used [2], while in [3] the temperature distribution with respect to a thickness is approximated by Legendre polynomials. In this case, the order of the system of resolving equations depends on the degree of the polynomial [2] or the number of layers [3]. In [4], a nonlinear law is proposed. However the aspects of a rational choice of hypothesis allowing a description of the internal thermal condition of multilayered systems are insufficiently developed.

Consider a plate composed of an arbitrary quantity \((k = 1, 2, \ldots, n)\) of orthotropic layers. Contact surfaces of the layers are determined by \(z\)-coordinates of \(a_{k-1}\) and \(a_k\) \((a_k > a_{k-1})\) counted off an arbitrarily chosen coordinate plane \(x_1x_2\) up to the lower and upper boundaries of a layer \(k\). The summation is to be made with respect to dummy indices \(j, p, \ldots\), but no summation is taken over \(i = 1, 2; k, m\). Partial derivatives of the coordinates are designated with commas on the level of subscripts, while a time derivative — by a point above the function. Superscripts, unlike exponents, are bracketed.

A heat conduction equation for the \(k\)-th layer is of the form [5]

\[
\rho \frac{\partial^2 T^{(k)}}{\partial t^2} - \sum_{j=1}^{3} \lambda^{(k)}_{j} \frac{\partial^2 T^{(k)}}{\partial x_j^2} = \sum_{j=1}^{3} \frac{\partial q^{(k)}}{\partial x_j},
\]

where \(\lambda^{(k)}_{p}\) are thermal conductivities in the direction of the coordinate axes \(x_1, x_2\) and \(x_3 = z\); \(c_v^{(k)}\) is the volumetric heat capacity. Between the plate layers, the following ideal thermal contact conditions are satisfied

\[
T^{(k-1)}|_{+} = T^{(k)}|_{-}, \quad \lambda^{(k-1)}_{3} T^{(k-1)}|_{+} = \lambda^{(k)}_{3} T^{(k)}|_{-}.
\]

Here signs "+" and "-" designate the upper and lower boundaries of layers, respectively.

On the plate faces \((z = a_p)\), the boundary conditions [5]:

of the first kind

\[
T^{(1)}(x_i, a_p, \tau) = T_{0i}(x_i, \tau); \quad T^{(2)}(x_i, a_p, \tau) = T_{n}(x_i, \tau);
\]

of the second kind

\[
\lambda^{(1)}_{3} T^{(1)}(x_i, a_p, \tau) = q^{(1)}_{0}(x_i, \tau); \quad \lambda^{(2)}_{3} T^{(2)}(x_i, a_p, \tau) = q^{(2)}_{n}(x_i, \tau);
\]

of the third kind

may be set. In the relations (3)-(5), $\alpha_p$, $T_c^P$, $q_p (p = 0, n)$ are heat transfer coefficients, ambient temperature, and heat fluxes on the plate faces, respectively. Boundary conditions, analogous to (3)-(5), may be also prescribed along the plane contour. The initial conditions for the differential equation (1) are

$$T^I (x_i, z, \tau) = \bar{T}_0 (x_i, z).$$

(6)

where $\bar{T}_0 (x_i, z)$ is the given function describing the temperature distribution over the plate at the initial moment of time $\tau = \tau_0$.

To reduce the three-dimensional heat conduction problem to the two-dimensional one based on the approach [4], a hypothesis on temperature distribution with respect to the plate thickness is built. Initially, it is assumed that the temperature distribution obeys the piecewise-linear law [1]

$$k_1 (x_i, \tau) = \bar{T}_0 (x_i, \tau) f(k) (z) (p = 1, 2).$$

(7)

where $k_1 (x_i, \tau)$ and $k_2 (x_i, \tau)$ are the temperatures on the plate faces:

$$f_2^{(h)} (z) = \int_{a_z}^{z} [\lambda_2^{(h)}]^{-1} dz; f_1^{(h)} (z) = 1 - f_2^{(h)} (z)$$

are the given functions of the normal.

Substitution of the law (7) into the l.h.s. of the one-dimensional unsteady-state heat conduction equation

$$\bar{T}^{(k)} (x_i, z, \tau) = \bar{T}_0 (x_i, \tau) f^{(k)} (z)$$

(9)

subsequent integration with regard for interlayer contact conditions (2) and conditions on the plate faces (3) as well as replacement of the derivatives of temperature with respect to time by new desired functions [$\bar{T}_0 (x_i, \tau) = \bar{\kappa}_3 (x_i, \tau) \bar{T}_n (x_i, \tau)$] have allowed the temperature distribution with respect to a stack thickness to be written in the form:

$$\bar{T}^{(k)} (x_i, z, \tau) = \bar{\kappa}_p (x_i, \tau) f^{(k)} (z).$$

(10)

Henceforth $p = 1, ..., 4$. The functions of the temperature distribution with respect to the thickness $f^{(k)} (z)$ (j = 3, 4) within each layer are the cubic parabolas and determined by the expressions

$$f_2^{(h)} (z) = \int_{a_z}^{z} [\lambda_2^{(h)}]^{-1} dz; f_1^{(h)} (z) = 1 - f_2^{(h)} (z)$$

(11)

Using a variation technique, with the hypothesis (10) taken into account, we have derived a system of differential heat conduction equations which are as follows in a matrix form

$$[D] \{\chi\} - [C] \{\chi\} = \{q\},$$

(12)

where $[D]$ is the matrix of differential operators, the elements of which are

$$d_{jk} = P_{j}^{(j-1)} (..., i_1) + P_{j}^{(j-1)} (..., i_2) \cdots P_{3}^{(j-1)} (..., i_1) (j = 1, ..., 4);$$

(13)

$$d_{i_1} = \alpha (..., i_1); d_{i_2} = \alpha (..., i_2);$$