INTERNAL WAVES OF FINITE AMPLITUDE

R. L. Kulyaev

Much research has been devoted to unsteady fluid flow with a free boundary. For example, Ovsyannikov [1] and Nalimov [2] have proven theorems on the existence and uniqueness of a solution, and a number of papers have proposed algorithms for numerical solution, based on various chain methods [3-6] or potential-theory methods [7-9]. In the present article we consider two-dimensional potential waves of finite amplitude on the interface between two heavy fluids of different densities. The initial problem is reduced to the Cauchy problem for a system of two integrodifferential equations. An algorithm for the numerical solution of this system is constructed, and the results of calculations are presented.

1. We consider the plane motion of two nonviscous, incompressible fluids of different densities in the field of gravity. The flow is assumed to be continuous in the whole plane, potential flow outside the interface line separating the fluids and periodic in the horizontal direction.

Let a Cartesian coordinate system $x$, $y$ move in the horizontal direction with velocity equal to one half the sum of the flow velocities infinitely far from the interface line $L$, and let the $y$ axis be directed vertically upwards (Fig. 1). In the upper ($D_0$) and lower ($D_2$) flow domains the fluid velocity $V = (V_x, V_y)$ satisfies the equations

$$\text{div} V = 0, \quad \text{curl} V = 0, \quad (x, y) \in D_n, \quad n = 1, 2$$

and the following boundary conditions: The perturbed flow velocity damps out as we become removed from the interface line

$$V(x, y, t) \to \begin{cases} (-v_\infty, 0), & y \to +\infty \\ (v_\infty, 0), & y \to -\infty \end{cases}$$

no fluid flows across the interface line

$$v_n \cdot v = v \cdot v, \quad n = 1, 2,$$

and the drop in the hydrodynamic pressure at the interface line obeys the Laplace law

$$\rho_1 - \rho_2 = \mu k.$$  

Here $t$ is time, $v_\infty = \text{const}$, $v$ is a unit normal to $L$, $v$ is the translation velocity of the line $L$, $v_n$ and $\rho_n$ are the limiting values of the velocity $V$ and pressure $p$, respectively, on approaching $L$ from the domain $D_n$, $\mu$ is the coefficient of surface tension, and $k$ is the curvature of the interface line, with $k < 0$ ($k > 0$) if the domain $D_2$ is convex (concave) in the neighborhood of the point in question.

The initial velocity field

$$V(x, y, 0) = V_0(x, y)$$

is assumed to be known and to satisfy conditions (1.1)-(1.3).

Insofar as the interface line $L(t)$ is not known beforehand, the problem as stated is nonlinear.

2. Let us derive the equations of motion of the wave surface $L$, assuming that the surface has no self-intersection points and that as functions of

---


©1976 Plenum Publishing Corporation, 227 West 17th Street, New York, N.Y. 10011. No part of this publication may be reproduced, stored in a retrieval system, or transmitted, in any form or by any means, electronic, mechanical, photocopying, microfilming, recording or otherwise, without written permission of the publisher. A copy of this article is available from the publisher for $15.00.
arc length as the coordinates of the wave surface and the velocity discontinuity \( v_1 - v_2 \) are continuously differentiable to some order (the required order of smoothness will be made more precise below in Sec. 4).

By virtue of (1.3) the discontinuity in the flow velocity at the line \( L \) satisfies the equation
\[
\nu - v_1 = \gamma \frac{\partial \nu}{\partial s},
\]
where the function \( \gamma \) is real, and \( \xi = \xi + i \eta \) is the complex coordinate of a point on the interface line. The solution of the associated Riemann boundary-value problem [10] enables us to represent the velocity field in the form
\[
\mathbf{V}(z, t) = \frac{1}{2\pi i} \int \frac{\gamma(s, t) \cot \frac{\eta}{\lambda} (z - \xi(s, t))}{s - z} ds,
\]
where \( \lambda \) is the wavelength, \( l \) is the length of the wave profile, \( z = x + iy \), and the positive direction of traversing the contour \( L \) is the direction for which the domain \( D_1 \) in Fig. 1 is on the left. The last equation describes the velocity field induced by a vortex surface with intensity \( \gamma \). Hence, in order to satisfy the condition (1.3) that no fluid flow across the interface, it suffices to take one half the sum of the boundary values of the flow velocity as the translation velocity of the interface, that is, \( v = (v_1 + v_2)/2 \). Then \( v \) is determined from the Sokhotskii-Plemelj formulas [10] by means of the following singular integral:
\[
\mathbf{v}(s, t) = \frac{1}{2\pi i} \int \frac{\gamma(s, t) \cot \frac{\eta}{\lambda} (z(s, t) - \xi(s, t))}{s - z} ds.
\]

We note two consequences of the last equation:
\[
\int_0^1 v_x(s, t) ds = 0,
\]
\[
\int_0^1 v_y(s, t) ds = 0,
\]
where \( v_x \) and \( v_y \) are, respectively, the components of the velocity \( \mathbf{v} \) tangent and normal to the wave profile (see Fig. 1).

In a coordinate system associated with an arbitrary point of the line \( L \) and moving with velocity \( \mathbf{v} \), the Cauchy-Lagrange integral of the equations of motion of the fluid has the form
\[
\frac{\rho_n}{p_n} + \frac{\delta \Phi_n}{\delta t} + \frac{\nu_n^2 - v_n^2}{2} + g \eta = F_n(t), \quad n = 1, 2.
\]
Here \( \rho_n \) is the density of the fluid; the differentiation \( \delta/\delta t \) is performed in the moving coordinate system, so that \( \mathbf{v} = (\delta \xi/\delta t, \delta \eta/\delta t) \); \( \Phi_n \) and \( \nu_n \) are the limiting values of the velocity potential and relative fluid velocity, respectively, on approaching \( L \) from the domain \( D_n \); \( g \) is the acceleration due to gravity; and \( F_n \) are arbitrary functions. Taking into account (2.1), (2.3), and the equations
\[
v_{x1} = -\frac{\tau}{\lambda}, \quad v_{x2} = \frac{\tau}{\lambda},
\]
\[
\Phi_n(s, t) = \Phi_n(0, t) + \int_0^s \left( v_x + (-1)^n \frac{\eta}{\lambda} \right) d\sigma, \quad n = 1, 2,
\]
where \( \tau \) is a unit tangent to the line \( L \) (see Fig. 1), we can obtain from (2.6) the following expression for the pressure drop at the wave profile:
\[
P_1(s) - P_2(s) = \frac{1}{2} \int_0^s \left( (\rho_1 + \rho_2) \gamma + 2(\rho_2 - \rho_1) v_1 \right) d\sigma + \frac{\rho_2 - \rho_1}{2} \left( 2g \eta + \frac{\eta^2}{4} - v_1^2 \right) + \chi(t).
\]
Here the initial reference point for the arc length moves with velocity \( \mathbf{v}(0, t) \), and \( \chi \) is some function depending on \( F_1(t), F_2(t), \Phi_1(0, t), \) and \( \Phi_2(0, t) \). Eliminating the function \( \chi \) in the last equation and introducing the dimensionless parameter
\[
R = (\rho_2 - \rho_1)/(\rho_1 + \rho_2),
\]
we write condition (1.4) in the following form:
\[
\left. \frac{\delta}{\delta \sigma} \left( \gamma + 2Rv_1 \right) \right|_{\sigma = 0} = \frac{2u_k}{\rho_1 + \rho_2} - R \left( 2g \eta + \frac{\eta^2}{4} - v_1^2 \right) \right|_{\sigma = 0}.
\]