RAYLEIGH STABILITY OF SHEAR FLOW IN RELAXING MEDIA

V. E. Nakoryakov, V. V. Sobolev, and O. Yu. Tsvelodub

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The stability of steady-state flow is considered in a medium with a nonlocal coupling between pressure and density. The equations for perturbations in such a medium are derived in the linear approximation. The results of numerical integration are given for shear motion. The stability of parallel layered flow in an inviscid homogeneous fluid has been studied for a hundred years. The mathematics for investigating an inviscid instability has been developed, and it has been given a physical interpretation. The first important results in flow stability of an incompressible fluid were obtained in the papers of Helmholtz, Rayleigh, and Kelvin [1] in the last century. Heisenberg [2] worked on this problem in the 1920's, and a series of interesting papers by Tollmien [3] appeared subsequently. Apparently one of the first problems in the stability of a compressible fluid was solved by Landau [4]. The first investigations on the boundary-layer stability of an ideal gas were carried out by Lees and Lin [5], and Dunn and Lin [6]. Mention should be made of a series of papers which have appeared quite recently [7-9]. In all the papers mentioned flow stability is investigated in the framework of classical single-phase hydrodynamics. Meanwhile, in recent years, the processes by which perturbations propagate in media with relaxation have been intensively studied [10-12].

1. Fundamental Equations

We consider the problem of the stability of relatively small perturbations in the steady-state flow of a fluid with the following equation of state:

\[ \delta p = c_0^2 \delta \rho + \beta d\delta \rho/dt + \kappa d^2 \delta \rho/dt^2, \] (1.1)

where \( \delta p \) and \( \delta \rho \) are small perturbations of pressure and density. We shall neglect the effect of viscosity not caused by internal processes.

Let the unperturbed flow be defined by the relations

\[ p_0 = \text{const}, \quad \text{div} \ V = 0, \quad V_x = V(y). \]

We shall assume that the flow parameters oscillate about these quantities

\[ v_x = V(y) + \bar{\omega}(x, y, z, t); \]
\[ v_y = \bar{\nu}(x, y, z, t); \]
\[ v_z = \bar{\nu}(x, y, z, t); \]
\[ p = p_0 + \bar{\omega}(x, y, z, t); \]
\[ \rho = p_0 + \bar{\rho}(x, y, z, t), \] (1.2)

where the sign \( \sim \) refers to the small fluctuating quantities. Writing the Euler equation for Eqs. (1.2) and neglecting quantities of the second order of smallness in the perturbations and their derivatives we obtain the following system of partial differential equations:

\[ \begin{align*}
\frac{\partial u}{\partial t} + V \frac{\partial u}{\partial x} + \frac{\partial V}{\partial x} &= - \frac{1}{\rho_0} \frac{\partial p}{\partial x}; \\
\frac{\partial v}{\partial t} + V \frac{\partial v}{\partial x} &= - \frac{1}{\rho_0} \frac{\partial p}{\partial y}; \\
\frac{\partial W}{\partial t} + V \frac{\partial W}{\partial x} &= - \frac{1}{\rho_0} \frac{\partial p}{\partial z}; \\
\frac{\partial \rho}{\partial t} + V \frac{\partial \rho}{\partial x} + \rho_0 \frac{\partial \rho}{\partial x} + c_0^2 \frac{\partial \rho}{\partial y} + \frac{\partial \rho}{\partial z} &= 0,
\end{align*} \]

(1.3)

where the prime denotes differentiation with respect to y. The equation of state (1.1) is rewritten in the form

\[ \tilde{p} = c_0^2 \rho + \beta \frac{\partial \rho}{\partial x} + \frac{\partial V}{\partial x} \tilde{p} + \frac{\partial V}{\partial y} \tilde{p} + \frac{\partial V}{\partial z} \tilde{p} + \frac{1}{\rho_0} \frac{\partial p}{\partial t} + \frac{V^2}{\rho_0} \frac{\partial \rho}{\partial x}. \]

(1.4)

The system (1.3), (1.4) has solutions which are exponential functions of the variables x, z, t. We shall look for these solutions in the form

\[ u = u(y) \exp[i \alpha x - c t], \quad v = v(y) \exp[i \beta y], \quad W = W(y) \exp[i \gamma z]. \]

(1.5)

In what follows we shall restrict ourselves to investigating perturbations which are periodic in the spatial coordinates x and z. Consequently, in this case \( \alpha \) and \( \gamma \) in Eq. (1.5) must be real, and \( c \) can be complex, \( c = c_r + ic_i \).

If \( \alpha c_i \) in Eq. (1.6) is greater than zero, then the perturbations Eqs. (1.5) increase with time. If \( \alpha c_i \) is less than zero, the perturbations are damped.

Substituting Eq. (1.5) in Eqs. (1.3), (1.4), we obtain a system of ordinary differential equations in y,

\[ \begin{align*}
\alpha (V - c) u + vV' &= - \left( \frac{\alpha}{\rho_0} \right) p; \\
\alpha (V - c) v &= - \left( \frac{1}{\rho_0} \right) p; \\
\alpha (V - c) W &= - \left( \frac{1}{\rho_0} \right) p; \\
\left( \frac{\alpha}{\rho_0} \right) (V - c) \rho &= \alpha \rho; \\
p &= c_0^2 \rho + \beta \rho (V - c) \rho - \alpha \gamma (V - c)^2 \rho.
\end{align*} \]

(1.7)

Let \( V_{\text{max}} \) by the characteristic velocity of the unperturbed flow, and \( h \) its characteristic dimension. We introduce the dimensionless quantities

\[ \begin{align*}
\bar{\rho} &= \rho / \rho_0, \\
\bar{p} &= p / \rho_0 V_{\text{max}}^2, \\
\bar{V} &= V / V_{\text{max}}, \\
\bar{u} &= u / V_{\text{max}}, \\
\bar{V} &= v / V_{\text{max}}^2, \\
\bar{\rho} &= \rho / h V_{\text{max}}, \\
\bar{x} &= x / h, \\
\bar{y} &= y / h, \\
\bar{z} &= z / h,
\end{align*} \]

(1.8)

where \( M \) is the Mach number.

Using Eq. (1.8) we rewrite Eq. (1.7) omitting the bar above the dimensionless quantities:

\[ \begin{align*}
\alpha (V - c) u + vV' &= - \alpha p, \\
\alpha (V - c) v &= - \beta p, \\
\alpha (V - c) W &= - \gamma p, \\
\alpha (V - c) \rho &= \alpha \rho; \\
p &= (1/M^2) [1 + \alpha \beta M(V - c) - \alpha M^2 (V - c)^2] p.
\end{align*} \]

(1.9)

Thus, the problem of the stability of steady-state flow reduces to finding the eigenvalues of \( c \) for the system (1.9) which satisfy the specific boundary conditions of the perturbation. It can be shown that the stability problem of three-dimensional perturbations is equivalent to the stability problem of two-dimensional perturbations with a smaller Mach number and a larger parameter \( \beta \). Thus, we can restrict the treatment to two-dimensional perturbations.