RESISTANCE OF A LIQUID TO THE MOTION OF A GAS BUBBLE COMPRESSED BY PARALLEL WALLS

Yu. I. Petukhov, N. G. Skorobogatov, and V. I. Sosunov

It is well known [1] that for Reynolds numbers of $N_{Re} < 1$ the hydrodynamic equations of an incompressible liquid incorporating the Stokes approximation agree closely with experiment in relation to the resistance coefficients of cylinders, spheres, and other bodies.

In this paper we shall apply the approximation in question to the motion of a gas bubble severely compressed by parallel plane walls (Fig. 1). The shape of the bubble is taken as almost cylindrical in view of the fact that, when the bubble is severely crushed, the difference $(r_0 - r_1)$ is much smaller than $r_1$ (the distance from the axis of the bubble to the wetting point) and $r_0$ (the distance from the axis to the meniscus). We may therefore approximately regard the bubble as cylindrical with a radius $r_0$.

If in the Navier-Stokes equations we reject the inertial terms, the equations for the steady-state case will take the form

$$
\frac{1}{\rho} \text{grad} p = v \Delta V
$$

In addition to this we have the continuity equation

$$
div V = 0
$$

Applying the operation div to both sides of (1), we obtain

$$
div \text{grad} p = 0
$$

The system of equations (1) and (2) determines the flow field in the Stokes approximation.

Let us introduce a rectangular coordinate system with its $x$ axis along the flow, its $z$ axis perpendicular to the plane of the plate, and its $y$ axis parallel to the plates.

A flow of viscous liquid approaches the cylinder from infinity (on the left) in the direction of the $x$ axis. We shall consider that the bubble remains stationary, i.e., its resistance is compensated by some external force.

We have to solve the system of equations (1) and (2) subject to the following conditions. On moving away from the bubble to infinity the flow degenerates into the well-known Poiseuille flow; the velocity component normal to the surface of the bubble is equal to zero at that surface. We assume that the force of surface tension is so great that the sections cut off from the bubble by planes parallel to the walls are almost circular.

Let us take as a linear scale the radius of the bubble $r_0$ and as a scale of velocity the velocity $V_0$ at infinity. For $\xi = x/r_0 \to \infty$ or $\eta = y/r_0 \to \infty$ only one velocity component will be different from zero.

\[ \phi = V/V_0 (u, v, o) \]

\[ u = 1 - \frac{r^2}{h^2} \quad \left( \frac{r}{h} \right) \]

Here 2h is the distance between the plates.

Let us introduce the polar coordinates R and \( \varphi \)

\[ \xi = r \cos \varphi, \quad \eta = r \sin \varphi \quad (r = R/r_0) \]

In these coordinates, Eq. (2) takes the form

\[ \frac{\partial^2 \phi}{\partial R^2} + \frac{1}{R} \frac{\partial \phi}{\partial R} + \frac{1}{R^2} \frac{\partial^2 \phi}{\partial \varphi^2} = 0 \quad \left( s = \frac{p}{\rho \nu^2} \right) \]

Here \( \pi \) is the dimensionless pressure. We shall solve Eq. (4) by the method of variable separation and confine attention to the first term of the resultant series

\[ \phi = a \left( r + \frac{b}{r} \right) \cos \varphi \]

where \( a \) and \( b \) are constants of integration which have to be determined. It is clear from the geometry of the flow and Eq. (5) that \( \partial \phi / \partial \xi = a \) as \( r \to \infty \). We have hitherto everywhere assumed that the pressure is independent of \( \xi \) and that the component of velocity in the \( \xi \) direction is zero. The components \( u \) and \( v \) depend on \( \xi \), and we derive the following equations for these from (1):

\[ \frac{\partial^2 u}{\partial R^2} + \frac{1}{R} \frac{\partial u}{\partial R} + \frac{1}{R^2} \frac{\partial^2 u}{\partial \varphi^2} = a \text{Re} \left( 1 - \frac{b}{r^2} \cos 2\varphi \right) \]

\[ \frac{\partial^2 v}{\partial R^2} + \frac{1}{R} \frac{\partial v}{\partial R} + \frac{1}{R^2} \frac{\partial^2 v}{\partial \varphi^2} = -a \text{Re} \left( \frac{b}{r^2} \sin 2\varphi \right) \]

The values of \( u \) and \( v \) should vanish at \( \xi = \pm h/r_0 \). Furthermore, the radial velocity \( u_r = u \cos \varphi + v \sin \varphi \) at \( r = 1 \) should also vanish. By considering (3) and (6) as \( r \to \infty \) we obtain the relation \( a \text{Re} = -2r_0^2/h^2 \).

Remembering the foregoing relationship, from (6) and (7) we obtain the solutions

\[ u = \left( 1 - \frac{r^2}{h^2} \right) \left( 1 - \frac{b}{r^2} \right) \cos 2\varphi, \quad v = \left( \frac{r^2}{h^2} \right) \left( 1 - \frac{b}{r^2} \right) \sin 2\varphi \]

From the two latter expressions we have for \( u_r \)

\[ u_r = \left( 1 - \frac{r^2}{h^2} \right) \left( 1 - \frac{b}{r^2} \right) \cos \varphi \]

It follows from the foregoing and also from the conditions on the surface of the cylinder that \( b = 1 \) and hence

\[ u_r = \left( 1 - \frac{r^2}{h^2} \right) \left( 1 - \frac{1}{r^2} \right) \cos \varphi \]

The surface of the cylinder has a "liquid" boundary, so that the tangential stress on the surface of the cylinder should be equal to zero. The solution here obtained only satisfies this condition approximately. However, in calculating the total force acting in the flow on the cylinder, we shall consider the tangential stress as being exactly zero, and the force to be determined will be the total force arising from all the stresses normal to the surface.

The normal force acting on an element of the cylinder [2] is

\[ F_p = -p + 2 \pi \frac{\partial U}{\partial R} \left( U = uV_o \right) \]

where \( U \) is the dimensional component of the velocity \( V \). Transforming to dimensional quantities in (8)

\[ U_R = u_rV_o = V_o \left( 1 - \frac{r^2}{h^2} \right) \left( 1 - \frac{r^2}{h^2} \right) \cos \varphi \]