FORMULATION OF THE PROBLEM IN THE THEORY OF GEOMETRICALLY NONLINEAR VISCOELASTICITY

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A variational formulation of the problem of geometrically nonlinear viscoelasticity is proposed here: the problem then reduces to a system of integrodifferential equations in functions of a single argument.

Several theories of viscoelasticity have been proposed recently, reflecting the physical nonlinearity of the behavior of a material in either a one-dimensional or in a multidimensional state of stress [1]. These theories are applicable to materials where nonlinear effects already occur at small strains (quantities are considered small when their quadratic terms may, within given accuracy limits, be regarded as negligible in comparison with their linear terms). The smallness of strains at which nonlinear effects begin to appear in physically nonlinear materials allows one to retain the nonlinear terms only in the equations describing the physical relations of viscoelasticity, while the Cauchy relations and the equilibrium equations of motion remain linear so that, consequently, the metricity of a deformed and of an undeformed body volume becomes the same; moreover, the Euler formulation and the Lagrange formulation concur here. This makes it feasible to successfully solve the problems in the theory of physical nonlinearity [1, 2].

There exist materials, however, for which the relations between stresses and strains are linear at small strains, with nonlinear effects appearing only at large strains (when their quadratic terms become comparable with their linear terms). The presence of large strains makes it necessary to retain the nonlinear terms in all equations of the theory of geometrical nonlinearity.* This poses the main difficulty in solving the geometrically nonlinear problems. Indeed, while all differential equations in the theory of geometrical nonlinearity must account for the metricity of the deformed body volume, this metricity is not known, and its determination constitutes the gist of the problem. This difficulty explains, in our view, why hardly any problem in the theory of geometrically nonlinear viscoelasticity has been solved so far.

This study deals with the formulation of quasistatic and dynamic boundary-value problems in the theory of geometrically nonlinear viscoelasticity.

We consider a viscoelastic body bounded by a surface $\Sigma$. On this body act body forces $\mu F(x, t)$ ($x = \{x_1, x_2, x_3\}$, on surface segments $\Sigma_u$ are given displacements $\mu u_0(x, t)$, and on the remaining surface segment $\Sigma_\sigma$ is given a distributed surface load $q(x, t) = \mu q^0(t)Q^\sigma(x)$. At instant $t = 0$ are given initial displacements $u(x, 0) = 0$ and initial velocities $\dot{u}(x, 0) = \mu u_0(x)$. We are to determine the displacements $u(x, t)$ of points in the body. Coordinates $x_i$ ($i = 1, 2, 3$) are understood to be Lagrangian coordinates, the system of coordinates being tied to the undeformed volume of the body.

We assume that the quantities $q$, $u_0$, $F$, and $v$ are of the following orders of magnitude: $q \sim E$, $u_0 \sim r$; $F \sim E/r$; $v \sim r/T$, with $E$ denoting the modulus of elasticity, $r$ denoting the characteristic dimension of the body, and $T$ denoting the vibration period. Consequently, the dimensionless positive parameter $\mu$, not necessarily small, will determine the intensity of the external load and will be of the same order of magnitude as the strains produced in the body. One object of this report and others to be published later is to establish the range of variations of parameter $\mu$ within which the problem thus formulated will have a solution. We

*We consider here the three-dimensional problem. Problems of geometrically nonlinear viscoelasticity of plates and shells have been dealt with in [3, 4].


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note that this parameter introduced here is not in any way related to the second Lamé parameter, which will also appear in the analysis.

We consider three variants of the problem.

In the quasistatic formulation one assumes the inertial forces appearing within the body to be negligibly smaller than the space derivatives of internal stresses. It is assumed, furthermore, that \( F = q = 0 \) and \( u_0 = 0 \), as well as \( \nabla_0 = 0 \) at time \( t < 0 \). In the case of damped vibrations in the body one also assumes that \( F = 0, q = 0, u_0 = 0 \) at time \( t < 0 \), followed by the initial conditions

\[
\mathbf{u}(x, 0) = 0; \quad \mathbf{u}(x, 0) = \mu v_0(x).
\]  

(1)

In the case of forced vibrations one assumes that functions \( F, q, u_0 \) are periodic in time and determine on the infinite interval of time \((-\infty, \infty)\).

The relation between displacements and strains is expressed as in [5],

\[
\tilde{\mathbf{\varepsilon}}_{ij} = \frac{1}{2} (\nabla_i u_j + \nabla_j u_i) + \frac{1}{2} g^{mn} \nabla_i u_m \nabla_j u_n; \quad i = 1, 2, 3,
\]  

(2)

where \( \tilde{\mathbf{\varepsilon}}_{ij}, \tilde{\mathbf{\varepsilon}}_{ij} \) are the covariant components of the Almansi-Hamel tensor and the Cauchy-Green tensor, \( u_i \) are the covariant components of the displacement vector, \( g^{mn} \) are the contravariant components of the metric tensor for an undeformed body volume, and \( \nabla \) is the symbol of covariant differentiation.

The relation between stresses and strains is expressed as:

\[
\tilde{T} = 2G \left( \tilde{\mathbf{g}} - \frac{1}{3} I_3 \tilde{\mathbf{g}} \right) + \tilde{B} \left( \tilde{\mathbf{g}} - \frac{1}{12} I_1^2 \tilde{\mathbf{g}} \right) - B \left( 2I_1 \tilde{\mathbf{g}} - \frac{1}{2} I_1^2 \tilde{\mathbf{g}} \right) + c \left( I_2 - \frac{1}{4} I_1^2 \right) \tilde{\mathbf{g}} - I_1 \tilde{\mathbf{g}} + 2\tilde{\mathbf{g}}^2,
\]  

(3)

where \( \tilde{T} \) is the stress tensor, \( \tilde{\mathbf{g}} \) is the metric tensor for a deformed body volume, \( \tilde{\mathbf{\varepsilon}}, I_1, \) and \( L_2 \) are the Almansi-Hamel strain tensor, its first invariant, and its second invariant, respectively, and \( B, G \) are the Volterra integral operators

\[
\tilde{B}_\Phi = B \left[ \Phi(t) - \int_{-\infty}^{t} T_b(t-s) \Phi(s) \, ds \right]; \quad \tilde{G}_\Phi = G \left[ \Phi(t) - \int_{-\infty}^{t} T_u(t-s) \Phi(s) \, ds \right].
\]  

(4)

According to the assumption, the orders of magnitude of the descendent terms in expression (4) are

\[
\int_0^\infty T_G(s) \, ds \sim \mu; \quad \int_0^\infty T_b(s) \, ds \sim \mu.
\]

Relation (3) has been derived from Signorini's quadratic law, the latter involving the Lamé parameters and a third parameter \( c \). In this law the Lamé parameters have been expressed in terms of shear modulus \( G \) and bulk modulus \( B \), whereupon the constants \( B \) and \( G \) in the linear terms in Signorini's law have been replaced by the integral operators (4). The quadratic terms in Signorini's law have been retained without change. The integral terms with the strains squared are of the order of \( \mu^2 \), i.e., of the order of strains cubed, and have thus been omitted. Instead of the equilibrium equations of motion, we apply here the principle of possible displacements, according to which the elementary work of all active forces acting on the body (also the forces of inertia) through the possible displacements of which satisfy the geometrical boundary conditions

\[
\delta \mathbf{u} = 0 \quad \text{on} \quad \Sigma_u
\]  

(5)

is equal to zero:

*This expression has been derived from that in [5] for the potential energy.