CALCULATION OF THERMAL STRESSES IN AN ORTHOTROPIC CYLINDER

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The present article describes the determination of thermal stresses in an orthotropic cylinder with an axially symmetric temperature field taking into account the change in the elasticity constants over the cylinder radius. A closed solution is given for constant elasticity coefficients.

There is a qualitative difference between the behavior of isotropic and anisotropic cylinders in an axially symmetric temperature field. It was first established by Bolotin and Boletina [1] that stresses develop in an anisotropic cylinder during uniform heating. More detailed investigation of this effect showed it to be associated with anisotropy of the linear temperature distribution and to be independent of the anisotropy of elastic properties [2].

In contrast to previous calculations for orthotropic cylinders (e.g., [3]), where heating was uniform and the elastic constants were functions of the radius, the present investigation proceeds from the first-order equation system obtained previously [4]. This system contains no derivatives of the initial parameters, which is convenient for practical calculations, since the temperatures and thermoelastic constants are usually given in tabular form. The calculation algorithm is for a cylinder free at the ends. This case is no less important for practical purposes than that considered in the literature [1, 3, 5], in which the cylinder is restrained at its ends (the axial displacement equals zero).

We will thus consider an infinite elastic orthotropic cylinder with an axially symmetric temperature field (Fig. 1).

The deformation-tensor components are expressed in terms of the radial \( u(r) \) and axial \( w(z) \) displacements:

\[
\begin{align*}
\varepsilon_r &= \frac{du}{dr}; & \varepsilon_\phi &= \frac{u}{r}; & \varepsilon_z &= \frac{dw}{dz} = c = \text{const.} \\
\end{align*}
\]

The deformations and stresses are related by the formulas [6]

\[
\begin{align*}
\varepsilon_r &= \frac{\sigma_r}{E_r} - \nu_{\theta r} \frac{\sigma_\theta}{E_\theta} - \nu_{rr} \frac{\sigma_r}{E_r} + \alpha_r T; & \varepsilon_\theta &= -\nu_{\theta r} \frac{\sigma_r}{E_r} + \frac{\sigma_\theta}{E_\theta} - \nu_{\theta \theta} \frac{\sigma_\theta}{E_\theta} + \alpha_\theta T; \\
\varepsilon_z &= -\nu_{r z} \frac{\sigma_r}{E_r} - \nu_{r z} \frac{\sigma_\phi}{E_\phi} + \frac{\sigma_z}{E_z} + \alpha_z T,
\end{align*}
\]

where \( E_r, E_\theta, E_z, \nu_{r \theta}, \ldots \) are the moduli of elasticity and Poisson coefficients; \( \alpha_r, \alpha_\theta, \alpha_z \) are the coefficients of linear expansion; and \( T \) is the temperature. In this case, it follows from the symmetry condition that

\[
\begin{align*}
\nu_{\theta r} &= \nu_{r \theta}, & \nu_{\theta \theta} &= \nu_{\theta \theta}, & \nu_{r z} &= \nu_{r z}, & \nu_{z r} &= \nu_{z r}.
\end{align*}
\]
In order to reduce the volume of the calculations, we designate the directions \( r, \varphi, \) and \( z \) by the subscripts 1, 2, and 3 respectively. Equation (2) yields

\[
\sigma_j = \sum_{n=1}^{3} A_{j,n} \hat{e}_n - \beta_j T \quad (j = 1, 2, 3),
\]

where

\[
A_{11} = \frac{E_1}{D} (1 - v_3 v_2); \quad A_{12} = A_{21} = \frac{E_1}{D} (v_1 v_2 + v_1 v_3); \quad A_{13} = A_{31} = \frac{E_1}{D} (v_1 v_3 + v_2 v_3); \quad A_{22} = \frac{E_2}{D} (1 - v_1 v_3); \quad A_{23} = A_{32} = \frac{E_2}{D} (v_2 v_3 + v_2 v_1); \quad A_{33} = \frac{E_3}{D} (1 - v_1 v_2); \quad D = 1 - 2v_1 v_2 v_3 - v_1 v_2 - v_3 v_2 - v_1 v_3; \quad \beta_j = \sum_{k=1}^{3} A_{j,k} \hat{a}_k.
\]

If we substitute the values of \( \sigma_1 \) and \( \sigma_2 \) from Eq. (4) into the equilibrium condition \((d\sigma_1/dr) + (1/r)(\sigma_1 - \sigma_2) = 0\), use of Eq. (1) enables us to obtain a differential equation for \( u(r) \):

\[
d^2 u \left( \frac{1}{r} \right) + \frac{1}{r} \left( \frac{dA_{12}}{dr} \right) u = \frac{1}{r} \left( A_{23} - A_{13} - \frac{dA_{13}}{dr} \right) \beta_1 T + \frac{1}{r} \left( \beta_1 - \beta_2 \right) T.
\]

If we introduce the variables \( u_1 = u; \quad u_2 = \sigma_1 = A_{11}(du/dr) + A_{12}(u/r) + A_{13} \beta_1 T, \) Eq. (5) reduces to two first-order equations:

\[
\frac{d u_1}{d r} = \left( A_{22} - A_{11} \right) \frac{u_1}{r^2} + \left( A_{23} - A_{13} \right) \frac{u_2}{r} + \left( A_{23} - A_{13} \right) \frac{\beta_1 + \beta_2}{r} T,
\]

\[
\frac{d u_2}{d r} = A_{11} \frac{u_1}{r} + A_{12} \frac{u_2}{r} + A_{13} \beta_1 - \beta_2 \frac{\beta_1 + \beta_2}{r} T.
\]

The boundary conditions for a hollow cylinder are

\[
u_2(a) = 0; \quad u_2(b) = 0,
\]

where \( a \) and \( b \) are the inside and outside radii respectively.

For a solid cylinder, Eq. (7a) is replaced by the condition of finite \( u_1 \) with \( r = 0 \). The constant \( \epsilon \) is determined from the condition of equilibrium along the \( z \) axis:

\[
2 \pi \int_a^b \sigma_3 d r = 0.
\]

With constant elasticity parameters, we find the following overall solution to Eq. (5):

\[
u = c_1 r^n + c_2 r^{n-1} + \frac{r^n}{2n} \int_0^r r^{1-n} \Phi d r - \frac{r^{1-n}}{2n} \int_0^r r^{n+1} \Phi d r - \frac{Ae}{1-n} r,
\]

where \( n = \sqrt{A_{22}/A_{11}}; \quad A = (A_{13} - A_{23})/A_{11}; \quad \Phi = (1/A_{11}) [\beta_1 (dT/dr) + (1/r) (\beta_1 - \beta_2) T] \).

Substituting the solution of Eq. (9) into Eq. (4) and determining the integration constants \( c_1 \) and \( c_2 \) and the constant \( \epsilon \), Eqs. (7) and (8) are used to find the stresses in a solid cylinder:

\[
\sigma_j(r) = \eta_j(r) - \frac{\xi_{ij}(r)}{\xi_{11}(b)} \eta_1(b) \quad (j = 1, 2, 3).
\]

Here

\[
\xi_{ij} = \lambda_i r^{n-1} - \frac{m_j 2 \lambda_{ij}}{m_s n + 1} b^{n-1}; \quad \eta_j(r) = f_j(r) + \frac{2m_j}{m_s b} \left[ - \int_0^b r f_j d r + \int_0^b r \beta_2 T d r \right] - \beta_j T.
\]