FAILURE OF ORIENTED MATERIALS UNDER TENSION

V. P. Tamuzh and P. V. Tikhomirov

A model of failure in oriented materials is proposed which takes account of orientation and failure proceeding simultaneously. The model offered enables a number of anomalous properties observed in the materials under consideration to be explained.

A mathematical model of the failure of oriented polymers was proposed in [1] in which it was suggested that the forces in the material are taken up by linear supporting elements. The macrostresses are obtained by averaging the tensional forces of the supporting elements over the sphere. These forces are proportional to the elastic deformations. It was suggested that the criterion of failure was the critical deformation and the failure stress was obtained with various different ratios of oriented material. In [2], on the assumption of a certain kinetic equation for the rupture of the stressed linkages, the failure kinetics of preliminarily oriented polymeric materials were obtained. A similar model was developed in [3]. In [4] a model of the failure of an oriented polymer was considered which took account of the two processes of orientation and the buildup of damage, with the assumption that both these processes take place independently of one another.

In the present work failure together with simultaneous orientation of the material is considered. Competition between these two processes accounts for the experimentally observed anomalous relations between the ultimate tensile strength and the deformation rate.

Let us take the same model of the material as in [1], i.e., in the form of randomly directed elastic linear elements. As a result of deformation these elements are oriented swivelling and slipping relative to one another.

1. We shall introduce the distribution density of the linear elements in the directions \( \rho(\theta, \phi) \). To do this we shall mark out a microvolume in the material including a sufficient number of elements. We shall mark out the direction \( z \) (Fig. 1) and around it a small solid angle \( d\omega \). The number of elements within this angle will be denoted by \( dn \) and the total number of elements in the microvolume by \( N \), so that

\[
\rho = \frac{dn}{Nd\omega},
\]

and

\[
\int \rho d\Omega = 1.
\]

For unoriented material the bond density \( \rho_0 \) is the same in all directions and it follows from (1) that \( \rho_0 = 1/2\pi \). Similarly, we shall introduce the local damageability \( \Pi(\theta, \phi) \) as the ratio of the elements which fail to the total of elements included in the solid angle \( d\omega \). It is obvious that \( 1 \geq \Pi \geq 0 \).

2. In connection with our problem, the continuity equation in Euler's variables [5] will be written as

\[
\frac{\partial \rho}{\partial t} + \text{div} (\rho \mathbf{v}) = 0.
\]
where $v = \omega \times r$; $\omega$ is the angular-velocity vector of the elements resulting from deformation; $r$ is the vector parallel to the axis of the element, and its components in spherical coordinates are $r$, $0$, $0$. However, in studying failure we should consider the failed and unfailed elements as a two-phase system with transition from the one phase into the other. Then, besides Eq. (2), for the whole system we should also examine the continuity equation for a single phase, for example, for the elements which have failed with density $\rho\Pi$.

In connection with our problem the continuity equation for the elements which have failed has the form

$$\frac{\partial (\rho\Pi)}{\partial \tau} + \text{div} (\rho\Pi v) = \rho \frac{\delta \Pi}{\delta \tau},$$

in which $\delta \Pi/\delta \tau$ is the formation rate of failed elements. Multiplying Eq. (2) by $\Pi$ and subtracting Eq. (3) we shall obtain the relation for $\Pi$:

$$\frac{\partial \Pi}{\partial \tau} + v \text{grad} \Pi = \frac{\delta \Pi}{\delta \tau}.$$  

Later on we will consider the tension in the direction of the $x_3$ axis (see Fig. 1). For this case $\rho$ and $\Pi$ will be independent of $\varphi$ and Eq. (4) will be rewritten in the form

$$\frac{\partial \Pi}{\partial t} + \frac{\partial \Pi}{\partial \vartheta} \omega = \frac{\delta \Pi}{\delta \tau}.$$  

3. We shall deduce geometrical relations for the distribution of the density of the elements with respect to deformation and the angular rotation speed of the elements with respect to the deformation rate.

First of all, we shall stipulate that full deformation is made up of elastic deformation $\varepsilon_e$ and the deformation associated with orientation which, for brevity, we shall call plastic deformation $\varepsilon_p$. Both plastic and elastic deformations are determined with respect to the undeformed body. We shall assume that plastic deformation takes place without a volume change, but with elastic deformation a linear relation exists between the material volume $V$ and the elastic deformation, referred to the plastically deformed condition:

$$V = V_0 \left(1 + \frac{K\varepsilon_e}{1 + \varepsilon_p}\right),$$

where $V_0$ is the initial volume of the material; $K$ is a constant involving Poisson’s ratio for small elastic deformations $\mu$ through the relation

$$K = 1 - 2\mu.$$  

Consider the deformation of a unit cube. According to Eq. (6) we have $(1 + \varepsilon)^2 = 1 + K\varepsilon_e/(1 + \varepsilon_p)$, whence it follows that the transverse dimension $l$ of the unit cube after deformation is equal to

$$l = \sqrt{\frac{1 + \varepsilon_p + K\varepsilon_e}{(1 + \varepsilon)(1 + \varepsilon_p)}}.$$  

Now we can proceed directly to deduction of the geometrical dependence of the density distribution of the elements on the deformation (Fig. 2).

Consider the displacement of the elements included in the solid angle $d\omega = \sin \theta d\theta d\varphi$ around the point $A$. As a result of deformation these elements fall within the solid angle $d\omega_1 = \sin \theta d\theta d\varphi$ around the point $A_1$. From this it follows that $\rho_0 \sin \theta d\theta d\varphi = \rho(\theta) \sin \theta d\theta d\varphi$. Let us remember that $\rho_0$ is the distribution density of the elements of the undeformed material and is equal to $1/2\pi$. It follows from the last relation that

$$\rho(\theta) = \rho_0 \frac{\sin \theta d\theta}{\sin \theta d\theta}.$$  

410