EXPANSION OF A SCALAR FUNCTION ON THE UNIT SPHERE $S^{n-1}$ IN TERMS OF TENSOR COMPONENTS

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It is shown that each $l$-th partial sum of the Fourier series for a scalar function on the unit sphere $S^{n-1}$ can be expressed as a polynomial in the components of two completely symmetric tensors of rank $l$ and $l-1$, respectively, provided that $l \geq 1$. It is proposed that this expansion should be used to describe the so-called limiting surfaces in second rank tensor spaces.

1. Questions concerned with the expansion of a scalar function on a unit sphere in an $n$-dimensional Euclidean space in the form of a Fourier series have been treated in detail in [1]. We shall briefly recount these points.

We shall introduce the notation:

$E_n$ is a real $n$-dimensional Euclidean space;
$SO(n)$ is the group of rotations in $E_n$;
$S^{n-1}$ is a unit sphere in $E_n$;
g is an element of the group $SO(n)$, $g \in SO(n)$;
x, $y$ are vectors in the space $E_n$;
$\langle x, y \rangle$ is a scalar product;
$r = \|x\|$ is the length of the vector $x$, $\|x\| = \langle x, x \rangle$;
$\{e_i\}$ is an orthonormal basis in $E_n$, $\langle e_i, e_j \rangle = \delta_{ij}$;
$l_{ij}(g)$ is an element of the rotation matrix $g$, $l_{ij}(g) = \langle e'_i, e'_j \rangle = \langle e_i, e_j \rangle = \langle g^{-1}e_i, e_j \rangle$;
$\xi$ is the unit vector $\xi = g^{-1}e_n = e'_n$;
x, $x'$ are the Cartesian coordinates of the vector $x = x_ie_i = x'_ie'_i$;
$\theta_i$ are the spherical coordinates on $S^{n-1}$;
x $\times$ $y$ is the tensor product of the vectors $x$ and $y$;
$C^2(S^{n-1})$ is the set of all those scalar functions on the unit sphere $S^{n-1}$ with integrable squares of the modulus.

The Cartesian and spherical coordinates are related to one another in the following manner:

\[
x_1 = r \sin \theta_{n-1} \sin \theta_1 \sin \theta_2 \sin \theta_3 \ldots \sin \theta_{n-2} \sin \theta_{n-1};
\]
\[
x_2 = r \sin \theta_{n-1} \sin \theta_1 \sin \theta_2 \cos \theta_3 \sin \theta_4 \ldots \sin \theta_{n-2} \cos \theta_{n-1};
\]
\[
\vdots
\]
\[
x_{n-1} = r \sin \theta_{n-1} \cos \theta_{n-2};
\]
\[
x_n = r \cos \theta_{n-1};
\]

$0 \leq r < +\infty$; $0 \leq \theta_i < 2\pi$, if $k \neq 1$; $\cos \theta_k = \frac{x_{k+1}}{r_{k+1}}$, $\sin \theta_k = \frac{r_k}{r_{k+1}}$;

$r = r_n$, where $r_n^2 = x_1^2 + \ldots + x_{n-1}^2 + (x'_1)^2 + \ldots + (x'_k)^2$. 


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Any function \( \hat{f}(\xi) \equiv \mathbb{V}(S^{n-1}) \) may be expanded in a Fourier series which converges in the mean

\[
\hat{f}(\xi) = \sum_{l=0}^{\infty} \sum_{K} a_{Kl} \Xi_{Kl}(\xi),
\]

(1.1)

where

\[
a_{Kl} = \int_{S^{n-1}} \hat{f}(\xi) \Xi_{Kl}(\xi) \, d\xi;
\]

(1.2)

The index \( K \) runs over all possible sequences of integers \( K = (k_1, \ldots, k_{n-3}, \pm k_{n-2}) \) which are such that \( l = k_0 \geq k_1 \geq \ldots \geq k_{n-2} \geq 0 \);

\[
d\xi = \frac{\Gamma\left(\frac{n}{2}\right)}{2\pi^{n/2}} \sin^{n-1} \theta_{n-1} \cdots \sin \theta_1 \cdots d\theta_{n-1};
\]

\[
\Xi_{Kl}(\xi) = A_{Kl} \prod_{j=0}^{n-3} \left( \frac{2}{n-2j+1} \right)^{k_{j+1}} \left( \cos \theta_{n-j-1} \right)^{k_{j+1}} \theta_{n-j-1}! \pi \theta_{n-j-1} \sin^{k_{j+1}} \theta_{n-j-1} \sin \theta_1 \sin \theta_2 \cdots \sin \theta_{n-1} \sin \theta_n;
\]

\[
(A_{Kl})^2 = \frac{1}{\Gamma\left(\frac{n}{2}\right)} \prod_{j=0}^{n-3} \frac{2^{k_{j+1}+n-j-4}(k_{j+1}+1)!}{\Gamma(k_{j+1}+n-j-2)} \frac{(n-j-2)!}{2^{k_{j+1}+n-j-4}+2} \frac{\Gamma\left(\frac{n-j-2}{2}\right)}{\Gamma\left(\frac{n-j-2}{2}+k_{j+1}\right)};
\]

\[
G_{m}^{p}(l) = \frac{2^{m} \Gamma(p+m+1)}{\Gamma(p)} \sum_{a=0}^{\infty} \frac{(-1)^{k_{m-k}}}{2^{k_{m}+n-m-1} \Gamma(p+m) \Gamma(p+m-1) \cdots (p+m-k) \Gamma(m-k)}
\]

is a Gegenbauer polynomial.

The functions \( \Xi_{Kl} \) can also be expressed in terms of the matrix elements \( l_{ni} = \langle \xi, e_i \rangle = x_i / r \). Omitting the elementary, but rather lengthy, details, we shall simply state the final result.

\[
\frac{1}{\left[ \frac{k_{n-2}+k_{n+1}}{2} \right]} \left[ k_{n-2} \right] \sum_{m=0}^{k_{n-3}} \sum_{a_{n-3}} \left( \begin{array}{c}
-2k_{n-2} \end{array} \right)_{a_{n-1}} \times
\]

\[
\frac{1}{(n-j-4+2k_{j+1}-2a_{j})!!} \prod_{j=0}^{n-3} \left( \begin{array}{c}
k_{n-2} \end{array} \right)_{p} \times \prod_{i=4}^{n} \frac{(-1)^{k_{n-2}}}{a_{n-3}+b_{n+1}+\cdots+b_{n, p-1}} \times
\]

\[
\frac{1}{(n-j-4+2k_{j+1}-2a_{j})!!} \prod_{j=0}^{n-3} \left( \begin{array}{c}
k_{n-2} \end{array} \right)_{p} \times \prod_{i=4}^{n} \frac{(-1)^{k_{n-2}}}{a_{n-3}+b_{n+1}+\cdots+b_{n, p-1}} \times
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\]

We shall suppose, in accordance with their definition, that \( (-1)^{1} = 0 = (-1)^{2} = 1 \).

2. It can be shown, in a manner completely analogous to that for the three-dimensional case [2], that

\[
T_{i_{1} \cdots i_{m}}^{(n)}(g_{0}) \equiv \mathbb{V}(SO(n)) \ni g_{0} \in SO(n), \quad T_{i_{1} \cdots i_{m}}^{(n)}(g_{0}) e_{i_{1}} \cdots e_{i_{m}} = T_{i_{1} \cdots i_{m}}^{(n)}(g_{0}) e_{i_{1}} \cdots e_{i_{m}},
\]

These are the components of an \( n \)-dimensional tensor \( T^{(n)} \) of rank \( m \) in the basis \( \{ e_{i} \} = \{ e_{i} \} \)

\[
T^{(n)} = T_{i_{1} \cdots i_{m}}^{(n)}(g_{0}) e_{i_{1}} \cdots e_{i_{m}}, \quad \text{where } e \text{ is the identity element of the group } SO(n),
\]

Summation is carried out from 1 to \( n \) over the indices which are repeated twice. Here [1]

\[
\int_{SO(n)} d(g_{0}) = A_{n} \prod_{i=1}^{n} \int \frac{sin \theta_1 \cdots sin \theta_n}{2 \pi^{n/2}} \sin \theta_1 \cdots sin \theta_n;
\]

\[
A_{n} = \frac{\Gamma\left(\frac{n}{2}\right)}{2\pi^{n/2}};
\]