EXPANSION OF A SCALAR FUNCTION ON THE UNIT SPHERE $S^{n-1}$ IN TERMS OF TENSOR COMPONENTS

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It is shown that each $l$-th partial sum of the Fourier series for a scalar function on the unit sphere $S^{n-1}$ can be expressed as a polynomial in the components of two completely symmetric tensors of rank $l$ and $l-1$, respectively, provided that $l \geq 1$. It is proposed that this expansion should be used to describe the so-called limiting surfaces in second rank tensor spaces.

1. Questions concerned with the expansion of a scalar function on a unit sphere in an $n$-dimensional Euclidean space in the form of a Fourier series have been treated in detail in [1]. We shall briefly recount these points.

We shall introduce the notation:

$E_n$ is a real $n$-dimensional Euclidean space;
$SO(n)$ is the group of rotations in $E_n$;
$S^{n-1}$ is a unit sphere in $E_n$;
g is an element of the group $SO(n)$, $g \in SO(n)$;
x, y are vectors in the space $E_n$;
$x, y >$ is a scalar product;
r = $\|x\|$ is the length of the vector $x$,
$\|x\| = <x, x>$;
$\{e'_1\}$ is an orthonormal basis in $E_n$, $<e'_i, e'_j> = \delta_{ij}$;
$l_{ij}(g)$ is an element of the rotation matrix $g$, $l_{ij}(g) = <e'_i, e'_j> = <e'_i, g e'_j> = <e'_i, e_j> = <g^{-1}e'_i, e_j>$;
$\xi$ is the unit vector $\xi = g^{-1}e_n = e'_n$;
x$_i$, $x'_i$ are the Cartesian coordinates of the vector $x = x_i e_i = x'_i e'_i$;
$\theta_1$ are the spherical coordinates on $S^{n-1}$;
x $\times$ y is the tensor product of the vectors $x$ and $y$;
$\mathcal{S}^2(S^{n-1})$ is the set of all those scalar functions on the unit sphere $S^{n-1}$ with integrable squares of the modulus.

The Cartesian and spherical coordinates are related to one another in the following manner:

\[
\begin{align*}
x_1 &= r \sin \theta_{n-1} \cdots \sin \theta_2 \sin \theta_1; \\
x_2 &= r \sin \theta_{n-1} \cdots \sin \theta_2 \cos \theta_1; \\
& \vdots \\
x_{n-1} &= r \sin \theta_{n-1} \cos \theta_{n-2}; \\
x_n &= r \cos \theta_{n-1};
\end{align*}
\]

$0 \leq r < +\infty; 0 \leq \theta_1 < 2\pi; 0 \leq \theta_k < \pi, \text{ if } k \neq 1; \cos \theta_k = \frac{x_{k+1}}{r_{k+1}}; \sin \theta_k = \frac{r_k}{r_{k+1}}$;

$r = r_n$, where $r^2 = x_1^2 + \ldots + x_n^2 - (x'_1)^2 - \ldots - (x'_n)^2$.
Any function \( j(\xi) \equiv \mathbb{B}^2(S^{n-1}) \) may be expanded in a Fourier series which converges in the mean

\[
j(\xi) = \sum_{l=0}^{\infty} \sum_{K} a_K \xi_l(\xi),
\]

where

\[
a_K = \int_{S^{n-1}} j(\xi) \xi_K^l(\xi) \, d\xi;
\]

The index \( K \) runs over all possible sequences of integers \( K = (k_1, \ldots, k_{n-3} \pm k_{n-2}) \) which are such that

\[l = k_0 \geq k_1 \geq \ldots \geq k_{n-2} \geq 0;
\]

\[
d\xi = \frac{\Gamma\left(\frac{n}{2}\right)}{2^{n/2}} \sin^{n-2} \theta_{n-1} \ldots \sin \theta_0 \, d\theta_1 \ldots d\theta_{n-1};
\]

\[
\xi_K^l(\xi) = A_K \prod_{j=0}^{n-3} C_{j-k_j+1}^j \left( \cos \theta_{n-1-J} \right) \sin^{k_j-j} \theta_{n-j-1} \quad \prod_{j=0}^{n-3} \sin \theta_{n-j-1} \sin \theta_{n-j-1} \quad \prod_{j=0}^{n-3} \sin \theta_{n-j-1} \sin \theta_{n-j-1} ;
\]

\[
(A_K)^2 = \frac{1}{\Gamma\left(\frac{n}{2}\right)} \prod_{j=0}^{n-3} \frac{2^{2j+1+n-j-4} (k_j-k_{j+1})! (n-j+2k_j-2) \Gamma\left(\frac{n-j-2}{2}+k_{j+1}\right)}{\Gamma(n+k_{j+1}+n-j-2) \Gamma(n+k_{j+1}+n-j-2)} ;
\]

\[
G_m(l) = \frac{2^m \Gamma(p+m+1)}{\Gamma(p)} \sum_{n=0}^{m-3} \frac{(-1)^n (2k_{m-2n})!}{2^{2k_{m-2n}} (p+m) (p+m-1) \ldots (p+m-k_{m-2n})!}
\]

is a Gegenbauer polynomial.

The functions \( \xi_K^l \) can also be expressed in terms of the matrix elements \( l_{m} = \langle \xi, e_i \rangle = x_i/r \). Omitting the elementary, but rather lengthy, details, we shall simply state the final result.

\[
\xi_K^l(\xi) = A_K \sum_{a_{n-s-1}} \sum_{b_{n-s}} \sum_{a_{n-s}} \left( \frac{a_{n-s}}{2} \right) \prod_{j=0}^{n-s} \prod_{r=1}^{n-s} (-1)^s a_n b_{n-s} \ldots b_{n-r+1}
\]

\[
\times \prod_{j=0}^{n-s} \prod_{r=1}^{n-s} (-1)^s a_n b_{n-s} \ldots b_{n-r+1} \prod_{r=1}^{n-s} (-1)^s a_n b_{n-s} \ldots b_{n-r+1} \prod_{r=1}^{n-s} (-1)^s a_n b_{n-s} \ldots b_{n-r+1}
\]

\[
\times \frac{(n-j-4+2k_{j+1})!}{2^s (k_j-k_{j+1}+2a_j)! (n-j-4+2k_{j+1})!} l_{n} l_{m+1-k_{m-2n-1}} ! l_{m+1-k_{m-2n-1}} ! l_{m+1-k_{m-2n-1}} ! l_{m+1-k_{m-2n-1}} ! l_{m+1-k_{m-2n-1}} !
\]

We shall suppose, in accordance with their definition, that \((-1)^n = 0 \neq 1 = 1\).

2. It can be shown, in a manner completely analogous to that for the three-dimensional case [2], that

\[
T_{i_1 \ldots i_m}^{(n)}(g_0) = \int_{SO(n)} \prod_{j=1}^{i_1} f_{i_1}^{(g_1)} \ldots \prod_{i_m} f_{i_m}^{(g_m)} (g_0) \, dg; \quad g_0 \in SO(n),
\]

These are the components of an \( n \)-dimensional tensor \( T^{(n)} \) of rank \( m \) in the basis \( \{e'_{i'}\} = \{g_0^{-1}e_i\} \)

\[
T^{(n)} = T_{i_1 \ldots i_m}^{(n)}(g_0) e_{i_1} \times \ldots \times e_{i_m} = T_{i_1 \ldots i_m}^{(n)}(e) e_{i_1} \times \ldots \times e_{i_m} ,
\]

where \( e \) is the identity element of the group \( SO(n) \).

Summation is carried out from 1 to \( n \) over the indices which are repeated twice. Here [1]

\[
dg = A_n \prod_{k=1}^{n-1} \prod_{j=1}^{k} \sin^{j-1} \theta_j \sin \theta_j ;
\]

where

\[
A_n = \frac{n}{\prod_{k=1}^{n-1} \Gamma\left(\frac{k}{2}\right)} .
\]