AVERAGED CHARACTERISTICS OF STRESSED LAMINATED MEDIA

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The article is concerned with the problem of calculating the average rigidity characteristics of an elastic laminated medium as functions of the initial stresses. The classical equations of an inhomogeneous elastic body with initial stresses are used as the starting ones, with the procedure of averaging (homogenization) applied to them. In the problem considered, the nonlinearity of the averaging procedure becomes very important and leads to a difference in the resulting working formulas from those derived in a classical manner.

It is noted in [1-3] that the averaging description of bodies with initial stresses requires the use of the method of averaging applied directly to the original inhomogeneous body. The use of a formula similar to that applied to homogeneous bodies (so-called "intermediate" averaging) generally gives an incorrect result. Works [1-3] were of a theoretical nature. In [3] it is noted that further progress in obtaining applied results can be made by considering specific structures.

The present work suggests an analysis of the problem of averaging for bodies of laminated structure. Such type of problems can be of interest for performing geophysical calculations, in particular, for taking into account the effect of initial stresses on the propagation of longwave vibrations. The results presented were partially reported in [4].

Statement of the Problem. Let us consider a laminated medium of periodic structure. The layers formed by homogeneous elastic materials are parallel to the $Ox_1x_2$ plane and have thickness $\varepsilon \ll 1$ (which is formalized as $\varepsilon \to 0$ [5]). We write the problem of elasticity theory for a body with initial stresses in the form [6]:

$$
[(a_{ijkl}(x_3/\varepsilon) + \sigma^*_{ijkl}(\bar{x}, x_3/\varepsilon) \delta_{ik}) u^e_{jl}]_{j} = \rho(x_3/\varepsilon) u_{tt}^e + f_i. 
$$

(1)

Here $u^e$ are the permutations; $a_{ijkl}$ and $\rho$ are the tensor of elastic constants and density, respectively; $\sigma^*_{ijkl}$ is the tensor of initial stresses; $f$ are the mass forces. The functions $a_{ijkl}(y_3)$, $\rho(y_3)$, $\sigma^*_{ijkl}(\bar{x}, y_3)$ are periodic in $y_3 = x_3/\varepsilon$ with period $m$ (where $m$ is the period of the structure of the considered body in dimensionless variables). We take the boundary conditions at the boundary of the region in the form:

$$
\vec{u}^e(\bar{x}, t) = 0
$$

(2)

As shown below, the basic effects in averaging are not associated with boundary conditions. The initial conditions

$$
\vec{u}^e(\bar{x}, 0) = \vec{u}_{tt}^e(\bar{x}, 0) = 0
$$

(3)

also do not influence the basic effects.

When $\varepsilon \to 0$, the inhomogeneous medium considered can be replaced by a certain homogeneous averaged medium [5] close in mechanical behavior to the original one. In the absence of initial stresses, the averaged body is described by the equations of elasticity theory with the so-called averaged elastic constants $A_{ijkl}(0)$ [5, 7]. In particular, these values are determined in experiments with macrospecimens of inhomogeneous media (i.e., specimens of size $1 >> \varepsilon$).

Let the initial stresses $\sigma_{ij}$ be determined by solution of the problem
\begin{equation}
\sigma_{ij}^* = G_l \rho (x_3/e), \quad \sigma_{ij} = a_{ijkl} (x_3/e) \nu^e_{k,l}
\end{equation}
with conditions (2) and (3). The averaging of problem (4), (2), and (3) has the form [5, 7]:
\begin{equation}
(A_{ijkl} (0) \nu_{k,l})_{,ij} = G_l \langle \rho \rangle
\end{equation}
with conditions (2) and (3). In this case, as shown, for example, in [7],
\begin{equation}
\langle \sigma_{ij}^* \rangle \to \sigma_{ij} = A_{ijkl} (0) \nu_{k,l},
\end{equation}
where $\langle \cdot \cdot \rangle = \frac{1}{m} \int_0^m dy_3$ is the period-mean of the structure; $\sigma_{ij}$ are the averaged stresses. One of the possible suggestions for averaging problem (1) is the application of the formula (used in [6] for taking into account initial stresses) to a body with elastic constants $A_{ijkl}(0)$ and initial stresses $\sigma_{ij}$ (so-called "intermediate" averaging). However, as follows from [1-3], in general this suggestion leads to an erroneous result and averaging of (1) leads to the equation
\begin{equation}
(A_{ijkl} (\sigma) \nu_{k,l})_{,ij} = \langle \rho \rangle u_{,tt} + f_i,
\end{equation}
where in general
\begin{equation}
A_{ijkl} (\sigma) \neq A_{ijkl} (0) + \sigma_{ij} \delta_{ik}.
\end{equation}

We note that the formula used in [6] is applicable to the original (actual) inhomogeneous body. The inapplicability of the formula of [6] to an averaged body agrees with its fictitiousness (an averaged homogeneous body does not exist in reality).

In subsequent parts of the present work the statements formulated are justified at the level of the construction of a formal asymptotic expansion, working formulas are derived for stressed laminated media, and these formulas are examined.

Construction of an Averaged Problem. Introducing
\begin{equation}
\phi_{ijkl} (x_3/e) = a_{ijkl} (x_3/e) + \sigma_{ij}^* (x_3/e) \delta_{ik},
\end{equation}
it is possible to write (1) in a form similar to that of a system of equations from elasticity theory. Let us write (1) in variational form
\begin{equation}
\int_0^T \int_Q \phi_{ijkl} \psi_{ijkl} \nu_{,ij} dxdt = \int_0^T \int_Q \rho \nu_{,tt} \varphi dxdt + \int_0^T \int_Q \varphi dxdt
\end{equation}
for any $\varphi \in D ([0, T], H^1(Q))$ (for the definition of spaces see [8]).

Let us introduce the two-scale expansion
\begin{equation}
\bar{u} = \sum_{k=0}^{\infty} e^k \bar{u}^{(k)} (x, y), \quad \bar{\varphi} = \sum_{k=0}^{\infty} e^k \varphi^{(k)} (x, y),
\end{equation}
where $y = x/e$ is a fast variable [7]. The functions of the variables $x, y$ are differentiated according to the rule [7]:
\begin{equation}
\frac{\partial}{\partial x_i} f (x, y) = \left( \frac{\partial}{\partial x_i} + e^{-1} \frac{\partial}{\partial y_i} \right) f (x, y).
\end{equation}
Substitution of (11), with account for (12), into (10) yields