THE DYNAMIC PROBLEM OF A CYLINDER WITH A SLOWLY CHANGING INTERNAL BOUNDARY

U. Tokhtarov, M. A. Koltunov, B. I. Morgunov, and I. E. Troyanovskii

The problem of a hollow, viscoelastic cylinder confined in a shell is considered, and it is shown that the internal diameter changes. The fluctuations arising on applying a variable, internal pressure are analyzed.

A solution is given of the dynamic problem of the vibrations of a long, hollow cylinder, the internal boundary of which slowly changes, so that

\[ r_0 \leq r(\tau) \leq b; \quad \tau = \varepsilon t; \quad \lambda = \text{const}, \]

where \( r_0, r(\tau), \) and \( b \) are the initial, internal, and external radii of the cylinder, respectively; \( \tau \) is the so-called "slow time"; \( \varepsilon > 0 \) is a small parameter. The cylinder is made from an incompressible viscoelastic material and is enclosed in an elastic shell of thickness \( h \), elastic modulus \( E_0 \), Possion ratio \( v \), the density of the material being \( \rho_0 \). The shell is axially symmetrical and is subjected to variable, internal pressures \( p(t) \). It is assumed that axial deformation of the cylinder is absent, i.e., \( e_z = 0 \).

The connection between the stresses and strains can be written in the form

\[ \sigma_x(r, t) = 2G \left[ \varepsilon_x(r, t) - \varepsilon F \int_0^t R(t-s)e_x(r, s)ds \right]; \quad \sigma_x = \sigma_x - \sigma. \]

where \( \sigma_x \) and \( \varepsilon_x \) are the main stress and strain components, respectively, in \( x = (r, \theta) \); \( G \) is the instantaneous elastic modulus of the second kind; \( \varepsilon \) is a small parameter; \( \sigma \) is the hydrostatic pressure;

\[ R(t) = \frac{A \exp(-\beta t)}{t^{1-\alpha}} \]

is the relaxation center; \( A, \beta, \alpha (0 < \alpha < 1) \) are parameters of the cylinder material, which are determined by the methods discussed in [2].

Let \( \varepsilon \int_0^t R(s)ds = \int_0^t R_1(s)ds \ll 1; A_1 = \varepsilon A \), which corresponds to the case of low viscosity (see [1, 6]).

In order to determine the stress-strain distribution of the cylinder the equation of motion is written in the form

\[ \frac{\partial \sigma_r}{\partial r} + \frac{\sigma_r - \sigma_\theta}{r} = \rho \frac{\partial^2 u}{\partial t^2}; \]

and the incompressibility condition is

\[ \frac{\partial u}{\partial r} + \frac{u}{r} = 0. \]

The Cauchy ratio is

\[ e_r = \frac{\partial u}{\partial r}; \quad e_\theta = \frac{u}{r}; \quad e_z = e_0 = e_0 = e_z = 0; \]


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The boundary conditions on the external and internal faces of the cylinders are

\[ \sigma_r \bigg|_{r=A(t)} = -p(t); \quad \sigma_r \bigg|_{r=b} = -h \left( \frac{\partial u}{\partial t} + \frac{E_0 u}{(1-v^2)b^2} \right) \]  

(7)

and the initial conditions are

\[ u(r, t) \bigg|_{t=0} = u_0(r_0); \quad r(t) \bigg|_{t=0} = r_0; \]
\[ \frac{\partial u}{\partial t} \bigg|_{t=0} = u_0(r_0); \quad \dot{r} \bigg|_{t=0} = \varepsilon R. \]  

(8)

Here \( p \) is the density of the cylinder material, \( u = u(r, t) \) is the radial displacement.

Integrating (5) we obtain

\[ u = \frac{y(t)}{r}, \]  

(9)

where \( y(t) \) is an arbitrary time function. With allowance for (6) and (9), from (2) we obtain

\[ \sigma_r - \sigma_0 = -4G \left[ \frac{y(t)}{r^2} - \int_0^t R(t-s) \frac{y(s)}{r^2} \, ds \right] \]  

(10)

Inserting (9) and (10) in (4), and integrating with respect to \( r \) over the limits \( r(r) \) up to \( b \), with allowance for (7), we obtain a system of equations of the form

\[ \begin{aligned}
\dot{y}(t) + \omega^2 y(t) &= P(r, t) + \varepsilon \Phi(r, t, y); \\
\dot{r} &= \varepsilon R ,
\end{aligned} \]  

(11)

where

\[ \Phi(r, t, y) = B(r) \left[ \int_0^t R(t-s) y(s) \, ds \right]; \]
\[ \omega^2 = \omega^2(r) = -\frac{E_0 r^2 + 2Gb^3}{L(r)(br)^2}; \quad \frac{E_0}{1-v^2} = \frac{E_0 h}{1-v^2} - 2Gb; \]
\[ P(r, t) = \frac{p(t)b}{L(r)}; \quad B(r) = \frac{2Gb(b^2-r^2)}{L(r)br^3}; \quad L(r) = p_0h + b p \ln \left( \frac{b}{r} \right). \]  

(12)

The system (11) must be solved on the assumption of the initial conditions

\[ y(0) = r_0 u_0(r_0); \quad \dot{y}(0) = r_0 u_0(r_0). \]  

(12')

We will consider certain special cases of the systems (11) and (12').

1. Let \( p(t) = 0 \), when from (11) it follows that:

\[ \begin{aligned}
\dot{y}(t) + \omega^2 y(t) &= \varepsilon \Phi(r, t, y); \\
\dot{r} &= \varepsilon R ,
\end{aligned} \]  

(11')

The system (11), by means of the known transformation [3]

\[ y(t) = F \cos \psi \]  

(13)

can be reduced to a system with rapidly rotating phase of the form

\[ \begin{aligned}
\dot{F} &= -\frac{\varepsilon}{\omega} \left[ FR \frac{\partial \omega}{\partial r} \sin \psi + \Phi(r, t, F, \psi) \right] \sin \psi; \\
\dot{\psi} &= \omega - \frac{\varepsilon}{F \omega} \left[ FR \frac{\partial \omega}{\partial r} \sin \psi + \Phi(r, t, F, \psi) \right] \cos \psi; \\
\dot{r} &= \varepsilon R,
\end{aligned} \]  

(14)

where \( F \) and \( \psi \) are new variables (\( F \) is the vibration amplitude, and \( \psi \) is the vibration phase), and