NONLINEARITY OF THE DYNAMIC PROPERTIES OF RUBBERS AT LARGE INHOMOGENEOUS COMPRESSIVE DEFORMATIONS

A. I. Lukomskaya, L. T. Kalinova, and S. I. Bukhov
UDC 678.43:539.383

The isothermal inhomogeneous harmonic compression has been calculated for a material with nonlinear elastic characteristics and a hyperbolic relaxation time distribution. The experimental data for rubbers satisfactorily confirm the theory. The proposed characteristics of the elasto-hysteresis properties of rubbers do not depend on the test piece geometry and are functions of the degree of deformation.

At large equilibrium deformations the stress-strain relation for rubber is nonlinear and satisfactorily defined by the Mooney-Rivlin elastic potential [1, 2]

\[ \Phi = C_1 (\lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 3) + C_2 (\lambda_1^2 \lambda_2^2 + \lambda_1^2 \lambda_3^2 + \lambda_2^2 \lambda_3^2 - 3), \]  

where \( C_1 \) and \( C_2 \) are elastic constants, and \( \lambda_i \) (\( i = 1, 2, 3 \)) are the principal degrees of deformation related to the principal strains \( \varepsilon_i \) (\( i = 1, 2, 3 \)) by the expressions

\[ \lambda_i = 1 + 2 \varepsilon_i, \quad (i = 1, 2, 3). \]  

A form of the elastic potential \( \Phi \) similar to (1) has been established experimentally, in tensile creep tests and dynamic simple shear tests [4, 5], and theoretically [6], except that in this case at a given stress the \( \lambda_i \) are functions of time [3, 6], while at a given dynamic strain the constants \( C_1 \) and \( C_2 \) must depend on the test frequency \( \omega \) and temperature \( T \).

Accordingly, we can write* the expression for the principal nonequilibrium stresses in the form

\[ \sigma_{i \varepsilon} = 2 \lambda i \varepsilon_i^2 [C_i + C_2 (\lambda_i \varepsilon_i^2 + \lambda_k \varepsilon_k^2)] + p, \]  

where

\[ \sigma_i = 2 \varepsilon_i \varepsilon_i^2 [C_i + C_2 (\lambda_i \varepsilon_i^2 + \lambda_k \varepsilon_k^2)] + p, \]  

*The state of stress was analyzed by A. I. Lukomskaya.

Since the principal axes coincide with the direction of the coordinate axes, the equilibrium displacement \( w \) along the \( z \) axis was found previously in [7].

We will employ the model of a viscoelastic solid shown in Fig. 2 and adopt the simplified hypothesis of plane sections. In this case, on the basis of calculations [7] for the harmonic regime with given displacement amplitude \( H - H_0 \), the displacement in the element \( C_\infty \), which determines the displacement distribution in the test piece, is

\[ w = \frac{(H - H_0) z_0 (1 - \cos \omega t)}{H_0 \left[ 1 + \frac{2}{\pi} \cos \left( \frac{z_0}{H_0} \frac{\pi}{2} \right) \right]^{n_0}}. \]  

where \( k_0 n_0 = 1 \); \( n_0 \) is a form factor depending on the ratios \( a_0 : H_0 \), \( b_0 : H_0 \) (when \( a_0 : H_0 \approx b_0 : H_0 \approx 2/3 - 1, n_0 \approx 1 \)).
The derivative of the displacement
\[
\left[ \frac{\partial \varphi}{\partial \varphi} \right]_{\varphi=0} = \frac{(H - H_0)(1 - \cos \omega t)}{H_0 \left(1 - \frac{2}{\pi}\right)} = \nu (1 - \cos \omega t) \quad \text{(6a)}
\]

We assume that the viscous response of the material is linear, i.e., the mechanical model describing the behavior of the viscoelastic material (see Fig. 2) is characterized by the system of viscous elements
\[
\eta_{in} = \frac{\sigma_{in}}{\dot{\varepsilon}_{in}} \quad (n = 1, 2, 3, 4 \ldots) \quad \text{(7)}
\]
The total stress \(\sigma_i\) represents the sum of the equilibrium
\[
\sigma_{i\infty} = C_{\infty} \lambda_i \omega^2 \quad (i = 1, 2, 3) \quad \text{(8a)}
\]
and nonequilibrium
\[
\sigma_{i\varepsilon_i\varepsilon_{in}} = C_T \lambda_i \omega^2 \quad \text{(8b)}
\]
stresses:
\[
\sigma_i = C_{\infty} \lambda_i \omega^2 + \int \int \int C_T \lambda_i \omega^2 d\tau + \rho, \quad \text{(8)}
\]
where in accordance with (3) and (4) it is assumed that
\[
C_{\infty} = 2C_{\varepsilon_i \varepsilon_{in}}, \quad C_T = C(\tau) = 2C_{\varepsilon_i \varepsilon_{in}}; \quad C_{\infty} = C_T = 0. \quad \text{In this model the stress acts in the direction of the principal strain axis}
\]
\[
\varepsilon_i = \varepsilon_{i\infty} - \varepsilon_{i\varepsilon_i\varepsilon_{in}}, \quad \text{(9)}
\]
From (2), (7), (8b), and (9) in the \(n\)-th element
\[
\frac{\lambda_i \varepsilon_{i\varepsilon_i\varepsilon_{in}}}{2\tau} = \lambda_i \frac{d\lambda_i}{dt} - \lambda_i \varepsilon_{i\varepsilon_i\varepsilon_{in}} \frac{d\lambda_i}{dt}, \quad \text{(10)}
\]
where the relaxation time
\[
\tau = \frac{\eta_{in}}{2C_T}. \quad \text{(10a)}
\]
Solving differential equation (10) for \(\lambda_{i\varepsilon_i\varepsilon_{in}}\), we find an expression for the principal nonequilibrium elastic strain of the \(n\)-th element in terms of its relaxation time \(\tau\) and the principal degree \(\lambda_i\) of total deformation:

\[
\kappa \lambda_{i\varepsilon_i\varepsilon_{in}}^2 = e^{-\frac{\tau}{\lambda_i}} \left[\text{const} + 2 \int e^{\frac{\tau}{\lambda_i}} \frac{d\lambda_i}{dt} \, dt\right]. \quad \text{(11)}
\]

Using (5a), (5b), (6a), and (8), from (11) we can find the principal stresses \(\sigma_i\) \((i = 1, 2, 3)\) in the steady-state period of deformation at \(e^{-\tau/\lambda_i} = 0\). In this case \(\sigma_1 = \sigma_2\) and it is assumed that the test is conducted at \(\omega = \text{const}\).

From the equilibrium conditions [8] and the boundary conditions of zero stress at the free edges of the test piece it follows that \(p = -\sigma_1 = -\sigma_2\). The load \(Q\) compressing the test piece can therefore be found as follows:
\[
Q = \int \int \int \sigma d\lambda x d\gamma. \quad \text{(12)}
\]
where \(a\) and \(b\) are the sides of the test piece during deformation.

If the distribution of relaxation times is described by a rectangular hyperbola, as found in [9, 10], then with the assumptions made in [10] and the relation
\[
C(\tau) \cdot d(\tau) = A \frac{dT}{\tau}, \quad \text{(13)}
\]
expression (12) takes the final form
\[
Q = \frac{a b}{1 + \nu(1 - \cos \omega t)} \times \left\{ C_{\infty} \left[1 + \nu(1 - \cos \omega t)\right] - \frac{C_{\infty}}{1 + \nu(1 - \cos \omega t)} \right\} +
\]
\[
+ \sin \omega t \left[ A \left( v + \frac{3}{2} \nu^2 - \nu^3 \right) \right] - \cos \omega t K \left[ A \left( v + \frac{3}{2} \nu^2 - \nu^3 \right) \right] + \sin 2\omega t
\]

<table>
<thead>
<tr>
<th>Degree of compression</th>
<th>(C_{\infty}) from (\varepsilon_{\infty})</th>
<th>(\lambda)</th>
<th>(C_{\varepsilon_i \varepsilon_{in}})</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.05</td>
<td>2.75</td>
<td>0.301</td>
<td>2.3</td>
</tr>
<tr>
<td>0.10</td>
<td>2.01</td>
<td>0.250</td>
<td>1.43</td>
</tr>
<tr>
<td>0.15</td>
<td>1.63</td>
<td>0.210</td>
<td>1.023</td>
</tr>
<tr>
<td>0.20</td>
<td>0.98</td>
<td>0.200</td>
<td>0.815</td>
</tr>
<tr>
<td>0.25</td>
<td>0.5</td>
<td>0.150</td>
<td>0.683</td>
</tr>
<tr>
<td>0.30</td>
<td>0.79</td>
<td>0.150</td>
<td>0.603</td>
</tr>
</tbody>
</table>