EFFECT OF A LOCAL LOAD ON AN ORTHOTROPIC GLASS-REINFORCED PLASTIC SHELL

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The deformation of a layered orthotropic cylindrical shell under a local normal load is investigated on the basis of equations that do not depend on the hypothesis of straight normals. The solutions of the analogous classical problems were analyzed in [3]; a solution based on equations that take transverse shear strains approximately into account was proposed in [4]. The high degree of variability of the state of stress created by local loads indicates that it is quite important to take transverse shear strains rigorously into account in problems of this class. An attempt is made to estimate the error introduced by the hypothesis of straight normals and to calculate the load leading to debonding of the shell.

Consider an orthotropic layered cylindrical shell simply supported along the curved edges and subjected to normal pressure. In terms of applied shell theory the state of stress is described by the following system of equations [1]:

**equilibrium equations**

\[
\begin{align*}
\frac{\partial \sigma_1}{\partial \alpha} + \frac{\partial \tau_1}{\partial \beta} + \frac{R}{h} \frac{\partial \tau_1}{\partial \gamma} &= 0; \\
\frac{\partial \sigma_2}{\partial \beta} + \frac{\partial \tau_2}{\partial \alpha} + \frac{R}{h} \frac{\partial \tau_2}{\partial \gamma} &= 0; \\
\sigma_2 - \frac{\partial \tau_1}{\partial \alpha} - \frac{\partial \tau_2}{\partial \beta} - \frac{R}{h} \frac{\partial \sigma_2}{\partial \gamma} &= 0,
\end{align*}
\]

(1) (2) (3)

where \( \alpha \) and \( \beta \) are the dimensionless axial and circumferential coordinates referred to the radius \( R \); \( \gamma \) is the normal coordinate referred to the thickness \( h \); \( \sigma_1, \sigma_2, \) and \( \sigma_3 \) are the axial, circumferential, and radial normal stresses; \( \tau_1 \) and \( \tau_2 \) are the transverse shear stresses acting in the plane of orthotropy; \( \tau_1 \) and \( \tau_2 \) are the transverse shear stresses acting in the axial and transverse sections of the shell;

**physical relations**

\[
\begin{align*}
\sigma_1 &= B_1 \left( \frac{\partial u}{\partial \alpha} + v_2 \left( \frac{\partial v}{\partial \beta} + w \right) \right); \\
\sigma_2 &= \frac{B_2}{R} \left( \frac{\partial v}{\partial \beta} + w + v_1 \frac{\partial u}{\partial \alpha} \right); \\
s &= \frac{G_{12}}{R} \left( \frac{\partial v}{\partial \alpha} + v_2 \frac{\partial u}{\partial \beta} \right); \\
B_{1,2} &= \frac{E_{1,2}}{1 - \nu_1 \nu_2}; \\
\tau_1 &= G \left( \frac{1}{h} \frac{\partial u}{\partial \gamma} + \frac{1}{R} \frac{\partial w}{\partial \alpha} \right); \\
\tau_2 &= G \left( \frac{1}{h} \frac{\partial v}{\partial \gamma} + \frac{1}{R} \frac{\partial w}{\partial \beta} \right),
\end{align*}
\]

(4) (5)

where \( u, v, \) and \( w \) are the axial and circumferential displacements; \( w \) is the deflection, which is assumed not to depend on the variable \( \gamma \); \( E_1 \) and \( E_2 \) are the moduli of elasticity in the axial and circumferential directions; \( G_{12} \) is the shear modulus in the plane of orthotropy; \( G \) is the transverse shear modulus.

It is possible to reduce system (1)-(5) to three equations of the following form

\[ \frac{1}{G}[L_1(v_1) + L_3(v_3)] + \frac{R^2}{h^2} \frac{\partial^2 v_1}{\partial \alpha^2} = \frac{1}{R} \left[ \frac{\partial}{\partial \alpha} L_1(w) + \frac{\partial}{\partial \beta} L_3(w) \right]; \]

\[ \frac{1}{G}[L_2(v_2) + L_3(v_3)] + \frac{R^2}{h^2} \frac{\partial^2 v_2}{\beta^2} = \frac{1}{R} \left[ \frac{\partial}{\partial \beta} L_2(w) + \frac{\partial}{\partial \alpha} L_3(w) \right]; \]

\[ \frac{1}{R} \frac{\partial^2 w}{\partial \sigma^2} = \int \left[ \frac{\partial}{\partial \alpha} L_4(v_1) + \frac{\partial}{\partial \beta} L_4(v_2) \right] d\gamma + \frac{R}{h} L_4(p). \]

Here \( \bar{p} = p_1 - p_2 \); the operators \( L_1, L_2, L_3, L_4 \) take the form

\[ L_1 = B_1 \frac{\partial^2}{\partial \alpha^2} + G_{12} \frac{\partial^2}{\partial \beta^2}; \quad L_2 = B_2 \frac{\partial^2}{\partial \beta^2} + G_{12} \frac{\partial^2}{\partial \alpha^2}; \]

\[ L_3 = (G_{12} + \nu B_1) \frac{\partial}{\partial \alpha \partial \beta}; \]

\[ L_4 = \frac{1}{E_2} \frac{\partial^4}{\partial \sigma^4} + \left( \frac{1}{G_{12}} - \frac{2\nu_1}{E_1} \right) \frac{\partial^4}{\partial \alpha^2 \partial \beta^2} + \frac{1}{E_1} \frac{\partial^4}{\partial \beta^4}. \]

The derivation of Eqs. (6)-(8) involves the use of the condition of incompressibility of a normal element and the static boundary conditions on the inner and outer surfaces of the shell:

\[ \begin{align*}
\text{at } \gamma = 0: & \quad v_1 = v_2 = 0; \quad \alpha_2 = -p_1; \\
\text{at } \gamma = 1: & \quad v_1 = v_2 = 0; \quad \alpha_2 = -p_2.
\end{align*} \]

\[ \text{(9)} \]

Since the ends of the shell are assumed to be hinged, we represent the unknown functions in the form of the following expansions:

\[ \begin{align*}
v &= \sum_i \sum_j w_{ij} \sin \lambda_i \alpha \cos j\beta; \\
\tau_1 &= \sum_i \sum_j \tau_{ij}(\gamma) \cos \lambda_i \alpha \cos j\beta; \\
\tau_2 &= \sum_i \sum_j \tau_{ij}(\gamma) \sin \lambda_i \alpha \sin j\beta; \\
\bar{p} &= \sum_i \sum_j \bar{p}_{ij} \sin \lambda_i \alpha \cos j\beta,
\end{align*} \]

where \( \lambda_i = \pi i / l; \) \( l \) is the length of the shell. Assuming that the pressure \( \bar{p} \) is uniformly distributed over a square area with center at the midpoint of the length of the shell and sides equal to \( 2Re \), we obtain

\[ \begin{align*}
\bar{p}_{00} &= \frac{4p_0}{\pi^2} \sin \frac{\pi i R}{l} \left( -1 \right)^{i+1} \left( i = 0, j = 1, 3, 5 \ldots; \right) \\
\bar{p}_{ij} &= \left( -1 \right)^{i+1} \frac{8p_0}{\pi^2} \sin \frac{\pi i R}{l} \sin j\epsilon \quad \left( i = 1, 2, 3 \ldots; \right) \\
& \quad \text{at } \gamma = 1, 3, 5 \ldots.
\end{align*} \]

\[ \text{(12)} \]

Introducing the resultant \( Q = 4pR^2 \epsilon^2 \) and passing to the limit in Eqs. (12) as \( \epsilon \to 0 \), we write the coefficients corresponding to expansion (11) for a concentrated load \( Q \),

\[ \begin{align*}
\bar{p}_{0j} &= \left( -1 \right)^{i+1} \frac{Q}{\pi R l} \quad \left( j = 0, i = 1, 3, 5 \ldots; \right) \\
\bar{p}_{ij} &= \left( -1 \right)^{i+1} \frac{2Q}{\pi R l} \quad \left( j = 1, 2, 3 \ldots, i = 1, 3, 5 \ldots. \right)
\end{align*} \]

\[ \text{(13)} \]