EFFECT OF ECCENTRIC REINFORCEMENT OF THE EDGE
OF AN OPENING IN AN ANISOTROPIC PLATE
ON THE STATE OF STRESS

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A general mathematical approach is proposed to the solution of problems of the elastic equilibrium of anisotropic plates with a nonsymmetrically reinforced edge. This approach is based on the representation of the boundary conditions of the problem in the form of contour integrals containing an arbitrary function holomorphic in the region of the plate. Diagrams characterizing the dependence of the stresses in an orthotropic plate with an elliptic opening on the eccentricity of the reinforcement are presented.

Let us consider a thin anisotropic (m+1)-connected plate of thickness 2h bounded by simple closed contours $L = L_1 + L_2 + \ldots + L_{m+1}$ that do not intersect. In the case of an unbounded region $S$ the outside contour $L_{m+1}$ is at infinity. At each point of the plate there is a plane of elastic symmetry parallel to the middle surface $xoy$. The edge of the plate $L_1$ is nonsymmetrically reinforced with an elastic rib (ring) of constant cross section, while to the rest of the edge $L' = L - L_1$, bending moments $m(s)$, shearing forces $P(s)$ and the stresses $N$ and $T$ (N, T are the normal and tangential components of the given stresses) are applied. The plane of the ring axis is offset with respect to the middle surface of the plate by an amount $\xi_0$ (eccentricity, Fig. 1).

Irrespective of the type of loading the plate experiences bending and a generalized plane state of stress. In the absence of loading on the ring the boundary conditions of the problem take the form [1, 2]:

\[
\begin{align*}
\int_{L'} F(t) d\gamma &= \pm \frac{g}{2h} \int_{L_1} \left[ (r_0 - r_1) \epsilon_0 + \eta_0 \frac{d\theta_0}{dt} \right] F''(t) dt + \int_{L'} (N + iT) F(t) dt; \\
\int_{L'} F(t) d\gamma &= \pm \frac{g}{2h} \int_{L_1} \left[ (r_0 - r_1) \epsilon_0 + \eta_0 \frac{d\theta_0}{dt} \right] F'(t) dt + \int_{L'} (N + iT) F(t) dt; \\
\int_{L'} F(t) d\gamma - \int_{L_1} F(t) d\gamma &= - \int_{L_1} \left[ \frac{r_0 - r_1}{r_0} \epsilon_0 + (r_1 - r_0) \frac{d\theta_0}{dt} + i\theta_0 - \gamma \frac{d\theta_0}{dt} \right] F(t) dt + \int_{L'} (N + iT) F(t) dt; \\
\int_{L'} F(t) d\gamma - \int_{L_1} F(t) d\gamma &= - \int_{L_1} \left[ \frac{r_0 - r_1}{r_0} \epsilon_0 + (r_1 - r_0) \frac{d\theta_0}{dt} + i\theta_0 - \gamma \frac{d\theta_0}{dt} \right] F(t) dt + \int_{L'} (N + iT) F(t) dt; \\
\int_{L'} F(t) d\gamma &= \pm \int_{L_1} \left[ \frac{r_1}{r_0} \epsilon_0 + (r_1 - r_0) \frac{d\theta_0}{dt} + i\theta_0 - \gamma \frac{d\theta_0}{dt} \right] F(t) dt + \int_{L'} (N + iT) F(t) dt; \\
\int_{L'} F(t) d\gamma &= \pm \int_{L_1} \left[ \frac{r_1}{r_0} \epsilon_0 + (r_1 - r_0) \frac{d\theta_0}{dt} + i\theta_0 - \gamma \frac{d\theta_0}{dt} \right] F(t) dt + \int_{L'} (N + iT) F(t) dt; \\
\int_{L'} F(t) d\gamma &= \pm \int_{L_1} \left[ \frac{r_1}{r_0} \epsilon_0 + (r_1 - r_0) \frac{d\theta_0}{dt} + i\theta_0 - \gamma \frac{d\theta_0}{dt} \right] F(t) dt + \int_{L'} (N + iT) F(t) dt; \\
\end{align*}
\]
Here we have employed the notation:

\[ U = \sum_{j=1}^{2} \left[ (1 + i s_j \varphi_j(z_j)) + (1 + i s_j \overline{\varphi_j(z_j)}) \right]; \]

\[ V = \sum_{j=1}^{2} \left[ (p_j + i q_j \varphi_j(z_j)) + (p_j + i q_j \overline{\varphi_j(z_j)}) \right]; \]

\[ U^* = \sum_{j=1}^{2} \left[ (1 + i m_j) \varphi_j^*(z_j^*) + (1 + i m_j) \overline{\varphi_j^*(z_j^*)} \right]; \]

\[ V^* = \sum_{j=1}^{2} \left[ \left( q_j^* + i \frac{p_j^*}{m_j} \right) \varphi_j^*(z_j^*) + \left( q_j^* + i \frac{p_j^*}{m_j} \right) \overline{\varphi_j^*(z_j^*)} \right]; \]

\[ I_j(t) = m_j(s) + \int_{0}^{s} P_j(s) ds + i C_j, \quad (j = 2, 3, \ldots, m + 1), \]

where, in their turn, \( s_j, \mu_j \) are the roots of the corresponding characteristic equations; \( p_j, q_j, p^*_j, q^*_j \) are known constants; \( \varphi_j(z_j) \) are the complex potentials of the variables \( z_j = x + s_j y \) of the generalized plane state of stress; \( \varphi_j^*(z_j^*) \) are the complex potentials of the variables \( z_j^* = x + m_j y \) of the flexural state of stress [2, 3]; \( F(z) \) is an arbitrary function of the variable \( z = x + iy \) holomorphic in the region of the plate \( S \); \( t \) is the affix of a point on the contour \( L_1 \); \( g = E'F \); \( A \) and \( C \) are the stiffnesses of the ring in tension, bending and torsion; \( r_1 \) is the radius of curvature of the extreme fiber of the ring welded to the plate; \( r_0 \) is the radius of curvature of the neutral (for pure bending) fiber of the ring \( L_0 \) at a distance \( \eta_0 \) from the central axis; \( r_2 \) is the radius of curvature of the extreme fiber not in contact with the plate; \( e_0 \) is the relative elongation of the fiber \( L_0 \); \( \theta_n, \theta_\tau, \theta_\nu \) are the angles of rotation of the cross section of the ring about the natural axes \( nrb \); \( \xi^* \) is the distance from the middle surface of the plate to the plane of the ring axis; \( C_j \) are real constants. The upper sign is taken at \( r_1 < r_0 \), the lower sign at \( r_1 > r_0 \).

Boundary conditions (1) serve for determining the complex potentials \( \varphi_j(z_j), \varphi_j^*(z_j^*) \) and the ring strain components \( \varphi_j, \theta_n, \theta_\tau, \theta_\nu \). In particular, from (1) with \( \xi^* = 0 \) we obtain the boundary conditions of the plane problem of bending of anisotropic plates with a reinforced edge.

2. As an example we will consider an infinite anisotropic plate with an elliptic opening whose edge is nonsymmetrically reinforced by an elastic ring of constant cross section. The reinforcing ring is free from external forces and the stresses and bending moments in parts of the plate remote from the opening are equal to: \( \sigma_x^\infty = p; \sigma_y^\infty = q; \tau_{xy}^\infty = r; M_x^\infty = M_1; M_y^\infty = M_2; H_{xy}^\infty = M_{12} \).

The region occupied by the plate and the corresponding regions of variation of \( z_j = x + s_j y; z_j^* = x + m_j y \) are conformally mapped onto the exterior of the unit circle \( y_1 \) by setting

\[ z = \omega(\xi); \quad z_j = \omega_j(\xi_j); \quad z_j^* = \omega_j^*(\xi_j^*), \quad (2) \]

where for contour points of \( \gamma_1 \) the variables \( \xi, \xi_j, \xi_j^* \) take the same value \( \sigma = e^{i\theta} \).

In the mapped region (2) in the absence of a load on the ring the boundary conditions (1) take the form \( r_1 > r_0 \):

\[ \int_{y_1} F_1(\alpha) dU = - \frac{g}{2h} \int_{y_1} \left[ (r_0 - r_1) \varphi + i \frac{\eta_1}{\omega'(\sigma)} \frac{d\varphi}{d\sigma} \right] dF_1(\alpha) \frac{d\sigma}{\omega'(\sigma)} - \frac{g}{2h} \int_{y_1} \left[ \frac{\omega'(\sigma)}{\varphi'}(\sigma) \frac{d\varphi}{d\sigma} \right] dF_1(\alpha) \frac{d\sigma}{\omega'(\sigma)}; \]

\[ \int_{y_1} F_1(\alpha) dV = \frac{g}{2h} \int_{y_1} \left[ (r_0 - r_1) \varphi + i \frac{\eta_1}{\omega'(\sigma)} \frac{d\varphi}{d\sigma} \right] dF_1(\alpha) \frac{d\sigma}{\omega'(\sigma)} - \frac{g}{2h} \int_{y_1} \left[ \frac{\omega'(\sigma)}{\varphi'}(\sigma) \frac{d\varphi}{d\sigma} \right] dF_1(\alpha) \frac{d\sigma}{\omega'(\sigma)}; \]

\[ \int_{y_1} F_1(\alpha) dV = \frac{g}{2h} \int_{y_1} \left[ \frac{r_0}{r_1} \varphi + i \frac{(r_1 - r_0) \sigma}{\omega'(\sigma)} \frac{d\varphi}{d\sigma} + i \theta_\nu - \frac{1}{r_2} \left( \frac{i \sigma}{\omega'(\sigma)} \frac{d\varphi}{d\sigma} \right) \right] dF_1(\alpha) \frac{d\sigma}{\omega'(\sigma)}; \]