Variational Method of Crack-Contour Location for
Three-Dimensional Problem with Unilateral Constraints

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Extremal properties are established for the solution of the problem of cohesionless normal-rupture crack formation: namely, that the true contour of a Christianovich crack corresponds to the maximum volume of the cavity. Examples of the application of this principle are considered.

Mathematical Model of Christianovich Crack in an Elastic Body

In an elastic space compressed at infinity by a uniform stress $\sigma$, acting perpendicularly to the plane $S : z = 0$, forces symmetric with respect to $S$ but in the opposite direction are applied, leading to the formation of normal rupture over a certain part of this plane of maximum tensile stress (breakdown). If the effect of the cohesive forces of the material over the $S$ plane may be neglected in comparison with the applied forces, the resulting check (slit) may be described using the Christianovich model [1-4], developed in the context of the mechanics problems of hot rocks (for an evaluation of the limits of applicability of this cohesionless approximation to applied problems, see [5]).

In formulating the problem, the scheme of [4] is followed. Suppose that two half spaces with identical elastic properties (which may vary over the depth) are pressed together by a uniformly distributed stress $\sigma_{zz} = -\sigma$ (Fig. 1). Identical but opposite loads $q(r)$ tend to break the contact between these half spaces (such loads acting on the contour of the developing slit also result, as is known, from the above-mentioned volume forces disrupting the material [6, 7]). The displacements $w(r, \sigma)$ of the contours of the plane slit developing in the body, of unknown shape $G_\sigma$ in plan, and the normal pressure on the half spaces composing the body $p(r, \sigma) = -\sigma_{zz} / S$ must satisfy on $S$ conditions in the form of alternating equalities and inequalities

$$\begin{align*}
p(r, \sigma) - q(r) &\geqslant \sigma, \quad w(r, \sigma) > 0, \quad r \in G_\sigma, \\
p(r, \sigma) &\geqslant q(r) - \sigma, \quad w(r, \sigma) = 0, \quad r \in S \setminus G_\sigma.
\end{align*}$$

(1)

Here and below, in view of the symmetry, the conditions are written only for the upper half space; $r = (x, y)$ is a point of $S$. There are no tangential stresses nor cohesive forces at $S$. The inequalities in the conditions of Eq. (1) (unilateral constraints) reflect the physically clear requirement of "nonoverlapping" of the slit edges and the absence of a resulting tensile stress on the continuation of the slit -- in the region of overlap of the half spaces (see also [8]). In the given formulation, this problem of the breakdown of an

elastie body not subjected to tensile stress over the $S$ plane is a particular case of the Sin'orin problem [9] (on the loosening of an elastic half space away from a rigid base in ideal contact). The condition in Eq. (1) is "undetermined" (ambiguous) in Sin'orin—Fikera terminology. The existence and uniqueness of solutions of the Sin'orin problem has been demonstrated in [9]; certain estimates for a homogeneous half space are given in [8]. Note that for the given shape of the crack in plan (the slit) $G_0$, with the contour $\Gamma$, the displacements of the crack edges $w(r, \Gamma)$ and the applied load $Q(r)$ are related by the pseudodifferential equation

$$\Lambda G w(r, \Gamma) = Q(r).$$

In particular, for a crack in a homogeneous space of Young's modulus $E$ and Poisson's ratio $\nu$, the operator $\Lambda G$ takes the form [6]

$$\Lambda G w(x, y) = -\frac{E}{4\pi (1-\nu^2)} \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \int_0^1 \frac{w(x', y')}{(x-x')^2 + (y-y')^2} dx' dy'. \quad (3)$$

An ideal representation of the expansion process of the Christianovich crack $G_0$ when the compressive pressure $\sigma$ decreases from $\sigma = \infty$ (in which case the crack is closed) to $\sigma = \sigma_0$ will now be considered. Let $\Gamma_0$ be the contours of the corresponding regions of opening of the crack, for which Eq. (1) is satisfied. Then the functions

$$w_i(r; \Gamma_0) = \frac{\partial w(r, \sigma)}{\partial \sigma}, \quad p_i(r; \Gamma_0) = -\frac{\partial p(r; \sigma)}{\partial \sigma} \quad (4)$$

are the values on $S$ of the solution of the mixed problem with the conditions

$$p_i(r; \Gamma_0) = 1, \ r \in G_0; \ w_i(r; \Gamma_0) = 0, \ r \in S \setminus G_0. \quad (5)$$

Thus, $w_i(r; \Gamma_0)$ is the displacement of points of the contour of crack $G_0$ in the given body under the action of unit load (the solution of the equation $\Lambda G \tilde{w_i}(r; \Gamma_0) = 1$). Such solutions, called "unit solutions," are widely used in linear rupture mechanics, where they are usually treated from the viewpoint of the problem of a given crack in a homogeneous field of tensile forces [6, 7]. The following representation of the displacement $w(r, \sigma)$ in the problem with the conditions in Eq. (1) in terms of the unit solutions for the slits $G_r$ with $\tau \gg \sigma$ is obtained from Eq. (5)

$$w(r, \sigma) = \int_0^\sigma w_i(r; \Gamma_0) \ d\tau. \quad (6)$$

Equation (6) implies the property of smooth closure of the edges of the Christianovich crack on its contour when $q(r) \gg -Q_0$: $w(r, \sigma) = O(\rho_n^{3/2})$ in the vicinity of the contour $\Gamma_0$ of the growing crack, where $\rho_n$ is the distance from the point $r \in G_0$ to $\Gamma_0$.

Extremal Properties of Cohesionless-Slit Contours

The solution of the given problem has an important extremal property.* Of all the possible cracks $G$ for a given load field $Q(x) = q(x) - \sigma$, the true contour $\Gamma_0$ of the Christianovich crack has the maximum volume of the resulting cavity

*This assertion was expressed in the form of the Barenblatt hypothesis in discussing [10].