METHOD OF DESCRIPTIVE REGULARIZATION IN INVERSE PROBLEMS

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We propose algorithms for descriptive regularization that account for a priori concepts of the qualitative structure of the sought functions. The use of these concepts that possess stabilizing properties ensures rather high accuracy and quality of approximate solutions with an insignificant expenditure of computational resources.

The ideas of descriptive regularization developed in [1, 2] are very promising in creating effective algorithms for numerical solution of incorrect inverse problems of heat conduction. A specific feature of the descriptive regularization method is account for a priori concepts of the qualitative structure of the sought functions (knowledge of the regions of fixed sign, monotonicity, convexity, etc.). The use of these concepts that possess stabilizing properties [3, 4] ensures uniform convergence of approximate solutions.

We consider the basic points of constructing algorithms for descriptive regularization using, as an example, a boundary-value inverse problem for the quasilinear Stefan problem that consists in determining the temperature distribution $u(x, t)$ in the region $Q = Q_1 \cup Q_2$, the front of phase transition $\xi(t)$ for $0 \leq t \leq T$, and the boundary conditions $v(t)$ for $0 \leq t \leq T$ from the conditions

$$
c(x, t, u) \frac{du}{dt} - (a(x, t, u) u_x)_x + b(x, t, u) u_x = f(x, t, u),
$$

$$
(x, t) \in Q_1 = \{0 < x < \xi(t), 0 < t \leq T\},
$$

$$
(x, t) \in Q_2 = \{\xi(t) < x < l, 0 < t \leq T\},
$$

(1)

$$
\left. u \right|_{x=0} = v(t), \quad 0 < t \leq T,
$$

(2)

$$
a(x, t, u) u_x \big|_{x=l} = p(t), \quad 0 < t \leq T,
$$

(3)

$$
\left. u \right|_{x=\xi(t)} = u^*(t), \quad 0 < t \leq T,
$$

(4)

$$
\gamma(x, t, u) \big|_{x=\xi(t)} = \left[ a(x, t, u) u_x \big|_{x=\xi(t)} + \chi(x, t, u) \big|_{x=\xi(t)} \right], \quad 0 < t \leq T,
$$

(5)

$$
\left. u \right|_{t=0} = \varphi(x), \quad 0 \leq x \leq l, \quad \left. \xi \right|_{t=0} = \eta_0
$$

(6)

and the additional condition at $x = l$

$$
\left. u \right|_{x=l} = g(t), \quad 0 \leq t \leq T.
$$

(7)

We assume that the additional information contains data on monotonicity and convexity regions of the sought boundary function and has the form

\[ v \in V_\mu, \quad V_\mu = \{ v \in V_R, \quad \mu(t) v(t) \geq 0, \quad 0 \leq t \leq T \} , \quad (8) \]

\[ v \in V_\nu, \quad V_\nu = \{ v \in V_R, \quad v(t) v_\nu(t) \geq 0, \quad 0 \leq t \leq T \} , \quad (9) \]

where \( \mu(t) \) and \( v(t) \) are parametric functions; \( \mu(t) = \text{sign} \, v(t) \); \( v(t) = \text{sign} \, v_\nu(t) \); and \( V_R \) is a set of boundary functions \( v(t) \):

\[ \| v \|_{L_2[0,T]} \leq R, \quad R = \text{const} > 0 . \]

The method of descriptive regularization of the inverse problem (1)-(7) based on account for the a priori constraints (8) and (9) reduces to the variational problem

\[ \inf_{v \in V} J(v), \quad J(v) = \| u \|_{x=t} - g \|_{L_2[0,T]}^2 , \]

in which \( V \) is a set of admissible boundary functions. Depending on the availability of one or another kind of a priori information, \( V = V_\mu, V = V_\nu, V = V_\mu \cap V_\nu, \) and \( u|_{x=t} \) is the spur of the solution of the direct Stefan problem (1)-(6) corresponding to \( v \in V \) at \( x = t \).

Numerical implementation of the descriptive regularization method entails the problem of nonlinear programming

\[ \min_{\hat{v} \in \hat{V}} I(\hat{v}), \quad I(\hat{v}) = \sum_{j=0}^{N} \rho_j (u_{Mj} - g_j)^2 , \quad (10) \]

where \( \hat{v} = (v_0, ..., v_N) \) is a net boundary function on the net \( \omega_\tau = \{ t_j, 0 = t_0 < ... < t_N = T, t_j - t_{j-1} = \tau \} \); \( \rho_j \) are coefficients of the quadrature formula; \( g_j = g(t_j), u_{ij} (i = 0, M, j = 0, N) \) is a solution of the difference analog of the direct Stefan problem (1)-(6) on the nets \( \omega_h \times \omega_\tau \) in the region \( Q, \omega_h = \{ x_i, 0 = x_0 < ... < x_M = l, x_i - x_{i-1} = h \} \); and \( \hat{V} \) is a set of admissible functions \( \hat{v} \) that comply with the constraints:

\[ \sum_{j=0}^{N} \rho_j v_j^2 \leq R^2 , \]

\[ \mu_j (v_{j+1} - v_j) \geq 0, \quad j = 0, N-1, \quad \mu_j = \mu(t_j) , \]

\[ v_j \left( \frac{v_{j+1} - v_j}{\tau_{j+1}} - \frac{v_j - v_{j-1}}{\tau_j} \right) \geq 0, \quad j = 1, N-1, \quad v_j = v(t_j) . \]

2°. The descriptive regularization algorithm devised in [5] relies on the method of projecting conjugate gradients for the numerical minimization (10)-(13), which has a sufficiently high rate of iteration convergence and permits structural features of the sought function to be revealed in a few steps. An iteration process of this method in the finite-dimensional analog \( L_2(\omega_\tau) \) of the space \( L_2[0, T] \) is constructed proceeding from the initial approximation \( \hat{v}^s (s = 0) \) by the equations

\[ \hat{v}^{s+1} = \mathcal{P}_\hat{V} (\hat{v}^s - \alpha_s \hat{r}^s), \quad s = 0, 1, ..., \]

\[ \hat{r}^0 = \text{grad}_{L_2} I(\hat{v}^0), \quad \hat{r}^s = \text{grad}_{L_2} I(\hat{v}^s) - \beta_s \hat{r}^{s-1}, \quad s = 1, 2, ... , \]

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