THERMAL RESONANCE IN THE HEAT CONDUCTION
PROBLEM WITH A HEAT SOURCE MOVING ALONG
A CLOSED CIRCUIT

A. A. Repin

We found the conditions of resonant increase in the amplitude of temperature oscillations of the points of
a thin heat-conducting rod in the shape of a closed ring with a variable-power heat source moving on its
surface.

In [1] it is shown that the transfer of a liquid coolant in a closed circulation circuit, containing an immovable
active zone with a periodically varying heat flux, involves a resonance phenomenon consisting of a sharp increase
in the amplitude of temperature oscillations for a definite relationship among the oscillation frequency of the heat
flux in the active zone, the rate of coolant transfer, and the circuit length. In effect, this phenomenon is similar to
the resonance in mechanical oscillatory systems with coincidence of the natural frequencies and the frequencies of
disturbing processes. In the problem treated in [1], the natural frequency depends on the convective term in the
heat transfer equation that defines the time of liquid circulation in the circuit and on the boundary condition of
circuit closure that is tantamount to periodicity of the solution with respect to the coordinate running along the
circuit.

It may be assumed that, when similar conditions hold, resonance can also occur in other heat problems,
including the problem of heat conduction of solids in the presence of moving sources with the heat release \( q(x-ut, t) \). However, here the problem must have specific features of its own. We now consider them, taking, as a case in
point, a thin heat-conducting rod of length \( L \) in the shape of a closed ring with a heat source moving on its surface.
The simplest realization of such a ring is a wheel rim, during whose motion over the bearing surface heat release
occurs at the contact point. With a small thickness of the ring, a one-dimensional formulation of the problem appears
to be possible:

\[
\frac{\partial T}{\partial t} - a \frac{\partial^2 T}{\partial x^2} = q(x - ut, t). \tag{1}
\]

We introduce new variables \( \xi = x-ut \) and \( \tau = t \), thereby converting to a coordinate system that is stationary
relative to the source, and then Eq. (1) yields the equation

\[
\frac{\partial T}{\partial \tau} - u \frac{\partial T}{\partial \xi} - a \frac{\partial^2 T}{\partial \xi^2} = q(\xi, \tau). \tag{2}
\]

This equation differs from that examined in [1] by the presence of the second derivative of \( T \) and the need to take
into account the effect of the thermal diffusivity, and it should be given special consideration.

With a nonstationary heat flux \( q(\xi, \tau) = q_s(\xi) + q_0(\xi) \exp (i\omega \tau) \), the solution to Eq. (2) is sought in the
form

\[
T(\xi, \tau) = T_s(\xi) + \theta(\xi) \exp (i\omega \tau). \tag{3}
\]
Substituting Eq. (3) into Eq. (2) and introducing the dimensionless quantity \( \xi = \frac{x}{L} \) (the symbol of dimensionlessness is hereinafter omitted) instead of \( x \), for the amplitude of the nonstationary temperature component we obtain the equation

\[
\frac{d^2 \vartheta}{d \xi^2} + 2 \sqrt{\alpha} \frac{d \vartheta}{d \xi} - i \beta \vartheta = -Q_0(\xi),
\]

(4)

where

\[
\alpha = \left( \frac{uL}{2a} \right)^2, \quad \beta = \frac{\omega L^2}{a}, \quad q_0 = \frac{q_0 L^2}{a}.
\]

To solve Eq. (4), as in [1], we use the condition of solution continuity along the closed circuit rather than boundary conditions; however, in contrast to [1], not only must the function \( \vartheta(\xi) \) be continuous but also its derivative \( \frac{d \vartheta}{d \xi} \). A general solution \( \vartheta(\xi) \) can be found for arbitrary functions \( Q_0(\xi) \). The present study, however, treats only a single heat flux related to the vicinity of the point \( \xi = \xi^* \):

\[
Q_0(\xi) = \begin{cases} 
0 & \text{for } 0 < \xi < \xi^*, \quad \xi^* < \xi < 1, \\
\frac{A}{\Delta \xi} & \text{for } \xi^* \leq \xi \leq \xi^* + \Delta \xi, \quad \Delta \xi \to 0.
\end{cases}
\]

(5)

In this case Eq. (4) admits a solution in elementary functions. Based on this solution and taking, without loss of generality, the point \( \xi = 0 \) as the observable one, after simple manipulations we derive the following expressions for the temperature amplitude:

\[
|\vartheta| = \frac{A}{2 \sqrt{\alpha^2 + \beta^2}} \left[ |C_1 \exp(-\lambda_1 \xi^*) + C_2 \exp(-\lambda_2 \xi^*)| + \\
+ |D_1 \exp(-\lambda_1 \xi^*) + D_2 \exp(-\lambda_2 \xi^*)| \right]^{1/2}.
\]

(6)

\[
C_k = \frac{(-1)^{k+1} \left[ \mu \cos \lambda \xi^* + (-1)^k \lambda \sin \lambda \xi^* \right] a_k - \left[ \mu \cos \lambda \xi^* + \mu (-1)^{k+1} \sin \lambda \xi^* \right] b_k}{1 - 2 \cos \lambda \exp(-\lambda_k) + \exp(-2\lambda_k)},
\]

(7)

\[
D_k = \frac{(-1)^k \left[ \mu \cos \lambda \xi^* + \mu (-1)^{k+1} \sin \lambda \xi^* \right] a_k + \left[ \mu \cos \lambda \xi^* + \lambda (-1)^k \sin \lambda \xi^* \right] b_k}{1 - 2 \cos \lambda \exp(-\lambda_k) + \exp(-2\lambda_k)},
\]

(8)

\[
\lambda_k = (-1)^{k+1} \mu - \sqrt{\alpha}, \quad k = 1, 2;
\]

\[
a_k = 1 - \cos \lambda \exp(-\lambda_k); \quad b_k = \sin \lambda \exp(-\lambda_k);
\]

\[
\mu = \sqrt{\left( \frac{\alpha^2 + \beta^2}{2} \right)^{1/2} + \alpha}, \quad \lambda = \sqrt{\left( \frac{\alpha^2 + \beta^2}{2} \right)^{1/2} - \alpha}.
\]

(9)

(10)

Using the above expressions we analyze the dependences of \( |\vartheta| \) on the velocity \( u \) of motion of the source, its position on the circuit \( \xi^* \), the oscillation frequency \( \omega \) of the heat flux, and the thermal diffusivity \( a \). These dependences are ultimately governed by the parameters \( \alpha \) and \( \beta \) in Eq. (4) or, accordingly, by the parameters \( \lambda \) and \( \mu \).