SOLUTION OF COEFFICIENT INVERSE PROBLEMS OF HEAT CONDUCTION WITH ACCOUNT OF A PRIORI INFORMATION ON THE VALUES OF THE SOUGHT FUNCTIONS

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An algorithm is proposed to numerically solve coefficient inverse problems of heat conduction accounting for inequalities and equalities imposed on the sought functions.

The efficiency of computational algorithms for solving incorrect inverse problems of heat transfer is largely dependent on the possibility of taking into account the available a priori information on the sought functions [1, 2]. Taking, as a case in point, the solution of a coefficient inverse problem of heat conduction, the current study analyzes iterative computational algorithms allowing for such a priori known properties of the temperature-dependent sought functions as their positivity and specified values of these functions at certain values of the argument. The consideration of a priori information of this kind necessitates the numerical solution of extremal problems with imposed inequalities or equalities.

We examine a one-dimensional heat transfer process, whose mathematical model has the form of a boundary-value problem for a homogeneous quasi-linear heat conduction equation

\[ C(T) \frac{\partial T}{\partial \tau} + \lambda(T) \frac{\partial^2 T}{\partial x^2} = q(x, \tau), \quad 0 \leq x \leq l, \quad 0 \leq \tau \leq \tau_m; \]  

\[ T(x, 0) = T_0(x), \quad 0 \leq x \leq l; \]  

\[ \alpha_1 \lambda(T(0, \tau)) \frac{\partial T(0, \tau)}{\partial x} + \beta_1 T(0, \tau) = q_1(\tau); \]  

\[ \alpha_2 \lambda(T(l, \tau)) \frac{\partial T(l, \tau)}{\partial x} + \beta_2 T(l, \tau) = q_2(\tau), \]

where \( \alpha_1, \beta_1, \alpha_2, \) and \( \beta_2 \) are parameters, which can be used to analyze boundary conditions of the first, second, or third kind. Let, at a certain number \( N \) of spatial points with coordinates \( x = X_i, i = 1, N \), the time dependences of the temperature be measured

\[ T_{mea}(X_i, \tau) = f_i(\tau), \quad i = 1, N. \]

The coefficient inverse problem of heat conduction consists of determining the functions \( C(T) \) or \( \lambda(T) \) from conditions (1)-(5). It is assumed in this case that the type of boundary conditions (3) and (4) as well as the number of heat sensors \( N \) satisfy the conditions assuring uniqueness of the analyzed inverse problem (see, for example, [3, 4]).

For simplicity of the subsequent presentation we consider the case of determining one characteristic \( T(0, \tau) \). Extending the analyzed algorithms to multiparametric inverse problems does not present any considerable difficulties, although it involves more cumbersome calculations.

In constructing the iterative algorithms to solve the coefficient inverse problems, wherein the sought characteristics are functions of the temperature, parametrization of unknown functions must be employed. For example, the temperature dependence of the thermal conductivity may be written approximately in the form

\[ \lambda(T) \approx \lambda_0 + \sum_{n=1}^{N} \phi_n(T) \lambda_n, \]

where \( \lambda_0 \) is the known temperature-independent value of the thermal conductivity, and \( \lambda_n \) are the sought coefficients.
\[ \lambda(T) = \sum_{k=1}^{m} \rho_k \varphi_k(T) \equiv \langle \vec{\rho}, \vec{\varphi} \rangle, \quad T \in [a, b], \]  

where \( \vec{\rho} = [\rho_1, \rho_2, \ldots, \rho_m]^T \) is the parametric vector; \( \rho \in \mathbb{R}^m \); \( \mathbb{R}^m \) is the m-dimensional Euclidean space; \( \vec{\varphi} = [\varphi_1(T), \varphi_2(T), \ldots, \varphi_m(T)]^T \) is the vector of the given basic functions; \( \langle \cdot, \cdot \rangle \) is the scalar product; and \( a \) and \( b \) are the minimal and the maximal values of the temperature, respectively.

The mathematical model (1)-(4) at the prescribed characteristic allows computation of the temperature at the points where the heat sensors are installed in accordance with the measurement procedure (5). Thereby \( \lambda(T) \) is transformed into the vector function \( f = \{f_i(\tau), \tau = 1, N\} \) resulting from the measurements. In approximation (7), the parametric vector \( \vec{\rho} \) is sought. In this case, using the condition of equality of the computed and measured temperatures, the inverse problem may be presented as an operator equation

\[ A\vec{p} = f, \quad \vec{p} \in \mathbb{R}^m, \quad f \in F, \]  

where the operator \( A \) is constructed on the basis of the model (1)-(4) taking the measurement procedure (5) into account; and \( F \) is the space of the measured functions, for which the space \( L_2 \) of the functions with an integrable square is generally utilized.

The characteristics sought and, hence, the basic functions \( \varphi_k(T), k = 1, m, \) in the parametric presentation of the form (6) must satisfy definite smoothness requirements. These requirements ensue from the conditions of differentiability of a residual functional [2]. For example, the temperature dependence of the thermal conductivity \( \lambda(T) \) must be a twice continuously differentiable function. Continuity of the first derivative is essential for the volumetric specific heat \( C(T) \). Therefore, the basic functions \( \varphi_k(T), k = 1, m, \) should be chosen with regard to the indicated requirements. The smoothness requirements are fulfilled, in particular, by cubic B-splines [5].

A salient feature of the inverse problems of heat transfer is their incorrectness. This feature most often manifests itself in the fact that minor errors on the right side of Eq. (7) may lead to great deviations in the solution. On conversion to the finite-dimensional approximation (6) the inverse operator \( A^{-1} \) in Eq. (7) becomes bounded but the inverse problem remains poorly conditioned. Special regularizing methods and algorithms [6] are needed to solve the incorrect inverse problems.

An iteration regularization method [2] proved to be highly efficient in solving various inverse problems of heat transfer. In this method an iteration sequence, minimizing the residual functional, is constructed to solve the inverse problem of the form (7)

\[ J(p) = \|A\vec{p} - f\|_2^2 = \sum_{i=1}^{N} \int_0^T \left[ T(X_i, \tau) - f_i(\tau) \right]^2 d\tau. \]  

Here, use is made of gradient methods of first-order optimization. Successive approximations are constructed by the formula

\[ \vec{p}^{s+1} = \vec{p}^s + \gamma_1 \vec{G}(\vec{J} \vec{p}^{(s)}), \quad s = 0, 1, \ldots, s^*, \]  

where \( s \) is the iteration number, \( \gamma_1 \) is the descent parameter, \( \vec{J} \vec{p} \) is the gradient of the functional (2.1.11) calculated in the space \( \mathbb{R}^m \), \( \vec{G}(\vec{J}) \) is the vector characterizing the employed optimization method, \( p^0 \) is the initial approximation specified a priori, and \( s^* \) is the number of the last iteration determined during the solution of the problem from the regularizing residual condition

\[ J(\vec{p}) \leq \delta^2 \]  

(\( \delta^2 \) is the prescribed measurement error computed in the metric of the space \( F \)). The descent parameter \( \gamma_1 \) is obtained from the condition

\[ \gamma_1 = \text{Arg min}_{\gamma_1 > 0} J(\vec{p}^1 + \gamma_1 \vec{G}(\vec{J}_p^{(1)})) \]  

by any familiar method, for example, "golden section" [7]. The gradient \( \vec{J}_p \) is calculated using the solution of a boundary-value problem for a conjugate variable [2].