STABILITY OF A SPHERICAL STAR SYSTEM

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The stability of a spherically symmetric aggregate of point gravitating particles relative to arbitrary small perturbations is studied. It is assumed that in the absence of perturbations the particles move along circular trajectories chaotically oriented in space so that the total moment of the aggregate is zero. Dimensions of the aggregate are large in comparison to the gravitational radius, and particle velocities are nonrelativistic. It is shown that there exist initial mass-density distributions unstable relative to any perturbations with the exception of radial and dipole perturbations. A general stability criterion is formulated, with the form \( \frac{d\Omega^2}{dr} > 0 \), where \( \Omega^2 = \frac{4\pi G}{r} \int_0^r \rho_0(r) r^2 dr \) is the aggregate mass density, and \( G \) is the gravitational constant. The dependence of the increment on \( \ell \), the perturbation harmonic number, is studied. In the case of weak inhomogeneity \( r \frac{d\Omega^2}{dr}/\Omega^2 \ll 1 \) the increment is maximum for quadrupole perturbations (\( \ell = 2 \)) and decreases monotonically with increase in \( \ell \). In the opposite case of high inhomogeneity \( r \frac{d\Omega^2}{dr}/\Omega^2 \gg 1 \), the increment increases with increase in \( \ell \). In the case of weak inhomogeneity the increment may be as small as desired. For high inhomogeneity, instability develops over a time period smaller than the period of revolution of an individual particle. For \( d\Omega^2/dr < 0 \) the system is stable. Consideration of system microstructure in this case leads to damping of macrooscillations (system "heating").

1. The trajectory of every particle is characterized by an angular velocity vector \( \Omega \)

\[
\Omega^2(r) = \frac{1}{r} \frac{d\Phi_0}{dr} = 4\pi Gr^{-3} \int_0^r \rho_0(r) r^2 dr
\]

Here \( \Phi_0 \) is the self-congruent gravitational potential satisfying the equation

\[
\Delta\Phi_0(r) = 4\pi G \rho_0(r)
\]

The dimensions of the aggregate \( R \) are assumed much greater than its gravitational radius \( r_g = 2MG/c^2 \), where \( M \) is the mass of the aggregate; \( c \) is the speed of light (\( R \gg 10r_g \)), i.e., the system obeys the laws of classical mechanics. No assumptions are made relative to the form of the density \( \rho_0(r) \).

Let the system undergo a small arbitrary perturbation. In [1], using the self-congruent field approximation an equation was obtained describing the natural system oscillations developing in such a case. The perturbed self-congruent potential was represented in the form of a superposition of spherical harmonics. In view of the linearity of the equations, the system was solved for an individual harmonic of the perturbed potential \( \phi_1(r, \theta, \varphi, t) = \chi_\ell^m(r, \omega)Y_m^\ell(\theta, \varphi)e^{-i\omega t} \). For the radial portion of each harmonic \( \chi_\ell^m(r, \omega) \) an equation was obtained,

\[
A_\ell(r, \omega) \frac{\partial^2 \chi_\ell^m(r, \omega)}{\partial r^2} + \left( \frac{\partial A_\ell(r, \omega)}{\partial r} + \frac{2A_\ell(r, \omega)}{r} \right) \frac{\partial \chi_\ell^m(r, \omega)}{\partial r} - \frac{B_\ell(r, \omega)}{r} \chi_\ell^m(r, \omega) = 0
\]


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Here $\chi_l(r, \omega)$ is the radial portion of the $l$-th harmonic of the perturbed self-congruent potential in Fourier representation

$$A_l(r, \omega) = 1 + \Omega^2 \sum_{s=-l}^l |P_{ls}(\pi/2)|^2 \frac{\omega^2 - \Omega^2 - \Omega^2}{\Omega^2}$$

$$B_l(r, \omega) = r^2 \frac{d}{dr} \left( \frac{\Omega^2}{r} \sum_{s=-l}^l |P_{ls}(\pi/2)|^2 \frac{2\pi \Omega}{\omega_s (\omega_s^2 - \Omega^2 - \Omega^2)} \right) +$$

$$+ \Omega^2 \sum_{s=-l}^l |P_{ls}(\pi/2)|^2 \left\{ \frac{4\pi \omega_s + \omega_s^2 (\omega_s^2 - \Omega^2 - \Omega^2)}{\omega_s (\omega_s^2 - \Omega^2 - \Omega^2)} + \frac{\omega_s^2}{\omega_s (\omega_s^2 - 2\Omega^2)} \right\} + l(l + 1)$$

where $P_{mn}(\theta)$ are generalized Legendre polynomials [2], in particular,

$$P_{ls}(\pi/2) = \left\{ \begin{array}{ll} 2\left( \frac{l+s}{2}, \frac{1-s}{2} \right), & \text{if } (l+s) \text{ is even} \\ 0, & \text{if } (l+s) \text{ is odd} \end{array} \right.$$ (1.6)

$$\sum_{s=-l}^l |P_{ls}(\pi/2)|^2 = 1$$ (1.7)

$$\omega_s = \omega - s\Omega, \quad \alpha_s = \sqrt{(l+s)(l-s+1)}$$ (1.8)

and $\rho_0 = \rho_0(r)$ is the perturbed mass density of the system.

In the present study the frequency spectrum of natural oscillations obtained from Eq. (1.3) will be studied. We will show that among the eigenfrequencies of the system there may be imaginary values if the condition $\Omega^2/\Omega^2 > 0$ is fulfilled. This criterion indicates that in the case of monotonic increase of the initial mass density to the edge of the aggregate the system is known beforehand to be unstable. The dependence of the instability increment on $l$, the perturbation harmonic number, will be studied. In the case of weak inhomogeneity $r(d\Omega^2/dr) << \Omega^2$ the increment is maximized at $l = 2$ and decreases monotonically with increase in $l$.

The stability criterion (Sec. 5) has the form $d\Omega^2/dr < 0$. Hence, it follows that if the density decreases monotonically toward the edge of the aggregate, then the system is unconditionally stable. We will also show that at $d\Omega^2/dr < 0$, the natural system oscillations decay with time. These results are valid only for perturbations under the influence of which individual particles are not displaced by a distance of the order of the mean interparticle distance during the course of a revolution about the center of the aggregate. In the opposite case, linear instability occurs (the density amplitude increases linearly with time).

2. We will consider the solution of the system oscillation equation. In [1] an expression was obtained for Fourier harmonics of the macroscopic material velocity of the perturbed system, defined as $W(r) = j(r)/\rho(r)$, where $j$ is the material flux density. The expression for the spherical component $W_s(r)$ of the velocity $W(r)$ has the form

$$W_s(r, \theta, \varphi, t) = V_0(r, \omega) T_{ls}(\pi/2 - \varphi, \theta, 0)$$ (2.1)

The coefficient $V_0(r, \omega)$ is determined by the radial portion of the perturbed self-congruent potential $\chi_l(r, \omega)$ through the formula

$$V_0(r, \omega) = -i \sum_{s=-l}^l \frac{\omega}{\omega_s^2 - \Omega^2 - \Omega^2} \mathcal{P}_{ls}(\pi/2)$$

$$\times T_{ls}(0, -\pi/2, -\pi/2) \left\{ \frac{\partial \chi_l(r, \omega)}{\partial r} - \frac{2\Omega(r)}{\omega} \chi_l(r, \omega) \right\}$$ (2.2)

where $\mathcal{P}_{ls}(\pi/2)$ is the unified Legendre polynomial; $\omega_s$ and $\Omega^2$ are defined by Eq. (1.8). From the definition of $\mathcal{P}_{ls}(\theta)$ we have $\mathcal{P}_{ls}(\pi/2) = 0$ if $l + s$ is odd; thus, the index $s$ in the sum (2.2) takes on values $s = -l, -l + 2, \ldots, l - 2, l$.}

The total macroscopic velocity corresponding to arbitrary initial conditions is written in the form of the Fourier integral of Eq. (2.1),