SOLUTION OF INTERNAL PROBLEMS OF AERODYNAMICS UNDER TRANSITIONAL CONDITIONS USING A MODEL KINETIC EQUATION

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A method is developed based on the use of a model kinetic equation with a shock frequency \( \omega = v/l \) (\( l \) is the mean length of the free flight path; \( v \) is the modulus of the molecular velocity). The method is tested on several classical problems.

1. Description of Method. We shall use as basis the model kinetic equation

\[
v \frac{\partial f}{\partial \tau} = \frac{1}{\tau (v)} (f_0 - f)
\]

Here \( v \) is the molecular velocity; \( \tau^{-1}(v) \) is the shock frequency, depending on the velocity; \( f_0 \) is the local-equilibrium distribution function. In the consideration of the internal flows of a rarefied gas, in the majority of cases we can limit ourselves within the framework of a linear approximation (i.e., with small Mach numbers and small temperature gradients).

Therefore, we take

\[
f_0(v, r) = N (2\pi m \theta)^{-\frac{3}{2}} \exp \left( -\frac{p^2}{2\theta} \right) \left( 1 + \frac{p}{\theta} \right)
\]

We divide all the particles into two sorts: primary particles which have just flown away from the wall and have not undergone even one collision; secondary particles, which have undergone at least one collision. We write the kinetic equations for each sort separately:

\[
v \frac{\partial f_1}{\partial \tau} = \frac{1}{\tau} (f_{01} - f_1) + \frac{1}{\tau} f_{01}, \quad v \frac{\partial f_2}{\partial \tau} = -\frac{1}{\tau} f_1
\]

Here \( f_1 \) and \( f_2 \) are respectively the distribution functions of the primary and secondary particles; \( f_{01} \) and \( f_{02} \) are local-equilibrium functions normalized respectively for the densities of the number of primary and secondary particles.

The necessity for such a separation is a result of the following: with collisions between secondary particles the momentum and the energy of any given element of the volume are retained; with collisions between primary and secondary particles, in each element of the volume, there appears a momentum and an energy brought in by the particle from that point of the surface from which it was emitted. Therefore, at each point of its volume a gas consisting of secondary particles has sources of energy and momentum formed by the flows of primary particles at the given point.

Let us make the form of the function \( \tau(v) \) definite. We take \( \tau(v) = l/v \). Here \( l \) is the mean length of the free-flight path; \( v \) is the molecular velocity. (Various means for selecting \( \tau(v) \) are discussed in [1].)

We write the laws of conservation for the secondary particles:

\[
\int dp_{12} v = \int dp_{22} v = \frac{2}{V \pi} N_2 v_0
\]

\[
\int dp_{12} v = \int dp_{22} v = \frac{8}{3V \pi} N_2 v_0 \theta, \quad v_0 = \left( \frac{20}{m} \right)^{\frac{1}{2}}
\]
With the given selection of \( \tau (v) \), Eqs. (1.2) can be rewritten as

\[
f_1(\xi v, \xi v) = f_1(\xi v, \xi v) e^{-\xi/\theta}
\]

(1.6)

\[
f_2(\xi v, \xi v + \xi \nu) = f_2(\xi v, \xi v) e^{-\xi/\theta} + \int d\xi' e^{-(\xi - \xi')/\theta} f_2(\xi v, \xi v + \xi' + \xi u')
\]

(1.7)

Here \( \xi = \sqrt{v/v_s} \), \( f_0 = f_{01} + f_{02} \); \( \xi r_s \) is the radius vector of a point on the surface; \( f_s \) is the distribution function of the particles flying away from the surface; we assume that this distribution is locally Maxwellian.

We now fix the point \( r = r_s + \xi s \) and substitute expression (1.7), consecutively into the equations of conservation (1.3)-(1.5):

\[
\frac{2}{V \pi} N v_0 = \frac{1}{l} \int \frac{d r'}{4 \pi l'^2} \left\{ \frac{2}{V \pi} N v_0 + 3 \nu_0 \right\}
\]

(1.8)

\[
\frac{8}{3 V \pi} Q v_0 = \frac{1}{l} \int \frac{d r'}{4 \pi l'^2} \left\{ \frac{8}{3 V \pi} N v_0 \nu_0 \nu + 3 \nu_0 \frac{N v_0^2}{2} \right\}
\]

(1.9)

\[
\frac{8}{V \pi} N v_0 \frac{v_0^2}{2} = \frac{1}{l} \int \frac{d r'}{4 \pi l'^2} \left\{ \frac{8}{V \pi} N v_0 \nu_0 + 3 \nu_0 \frac{N v_0^2}{2} \right\}
\]

(1.10)

Here

\[
\nu = \frac{r - r'}{|r - r'|}, \quad R = |r - r'|, \quad Q = Nu
\]

Equations (1.8)-(1.10) express the density, the flows, and the pressure of the secondary particles in terms of the total values of these quantities over the whole volume. We now find the contribution of the primary particles. Integrating Eq. (1.6) over the space of the momenta at the point \( r = r_s + \xi s \), we find

\[
N_1 (r) = -\int d S' \xi e^{-R l} \left\{ N_s + \frac{4}{V \pi} N_s \frac{u_s}{v_0} \right\}
\]

(1.11)

\[
Q_1 (r) = -\int d S' \xi e^{-R l} \left\{ \frac{2}{V \pi} v_0 N_s \xi + 3 \xi (N_s, u) \right\}
\]

(1.12)

\[
N_1 \frac{v_0^2}{2} = -\int d S' \xi e^{-R l} \left\{ N_s \frac{v_0^2}{2} + \frac{8}{3 V \pi} v_0 (N_s, u) \right\}
\]

(1.13)

Here \( v_s \) is the velocity of the wall; \( v_0 = (2 \theta / m)^{1/2} \); \( \theta_s \) is the temperature of the wall.

The total density, flows, and pressure are expressed by the formulas

\[
N = \frac{4}{l v_0} \int \frac{d r'}{4 \pi l'^2} \left\{ N v_0 + \frac{3 V \pi}{2} \xi \nu_0 \right\} - \int d S' \xi e^{-R l} \left\{ N_s + \frac{u_s}{v_0} N_s \right\}
\]

(1.14)

\[
Q = \frac{4}{l v_0} \int \frac{d r'}{4 \pi l'^2} \left\{ 3 \nu_0 (\xi \nu + \frac{9 V \pi}{8} \xi - \frac{N v_0^2}{2} ) \right\} - \int d S' \xi e^{-R l} \left\{ \frac{2}{V \pi} N_s \nu_0 \xi + 3 \xi (N_s, u) \right\}
\]

(1.15)

\[
N \frac{v_0^2}{2} = \frac{4}{l v_0} \int \frac{d r'}{4 \pi l'^2} \left\{ N \frac{v_0^2}{2} + \frac{15 V \pi}{32} v_0^2 \xi \nu_0 \right\} - \int d S' \xi e^{-R l} \left\{ N_s \frac{v_0^2}{2} + \frac{8}{3 V \pi} v_0 (N_s, u) \right\}
\]

(1.16)

The parameters \( N_s \) and \( Q_s \) are determined from the condition of nonflow

\[
Q (r_o) v (r_o) = 0
\]

The system of equations (1.14)-(1.17) is a closed system of integral equations, sufficient in principal for the solution of any given problem involving the flows of a rarefied gas.

We consider below a number of problems whose solutions are well known; we shall use these as examples to demonstrate the correctness and the very high efficiency of the method developed here.

2. Couette Flow. Let there be two infinite flat plates, moving parallel one to the other at velocities of \( \pm u_s \). The distance between the plates is equal to \( 2a \). There is sought the flow density of the particles along the axis of the plates (Fig. 1).