SELF-SIMILAR PROBLEM OF THE DECAY OF
A TWO-DIMENSIONAL DISCONTINUITY

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The plane problem of the decay of an arbitrary two-dimensional discontinuity for the gasdynamics equations is considered. The initial surface of the discontinuity is assumed to have the shape of an angle close to $\pi$. The existence and uniqueness of the solutions of the problem in a linear formulation are proved.

Linear problems on the diffraction and reflection of shocks have been considered in [1-3]. The problem of the decay of a two-dimensional discontinuity reduces to a new boundary-value problem for equations of mixed type with discontinuous coefficients.

1. FORMULATION OF THE PROBLEM

Let some curve $\Gamma$ separate a plane into two parts, $D_0, D_1$. Two polytropic gases in states characterized by the constant parameters

\begin{align}
  u &= u_1, \quad v = v_1, \quad p = p_1, \quad \rho = \rho_1, \quad S = S_1, \quad \gamma = \gamma_1, \quad (x, y) \subseteq D_1 \\
  u &= u_0, \quad v = v_0, \quad p = p_0, \quad \rho = \rho_0, \quad S = S_0, \quad \gamma = \gamma_0, \quad (x, y) \subseteq D_0
\end{align}

(1.1)

are in $D_0$ and $D_1$ at the time $t = 0$.

The baffle $\Gamma$ vanishes at the time $t = 0$. It is required to describe the gas motion.

The solution of this problem is known in the particular case when the initial surface of discontinuity is a straight line. Here the solution of the problem of the decay of the discontinuity can be constructed in the class of self-similar solutions of the one-dimensional gasdynamics equations. It is evidently impossible to construct the solution of the problem in the class of self-similar solutions for an arbitrary curve $\Gamma$.

The necessary condition for self-similarity is invariance of the initial data of the problem relative to the transformation of the independent variables $x, y$ corresponding to the infinitesimal operator [4]

$$
x \partial / \partial x + y \partial / \partial y
$$

This condition is satisfied if and only if the initial surface of discontinuity $\Gamma$ has the shape of an angle. In this case the problem of seeking the self-similar solution describing the two-dimensional decay of a discontinuity originates.

The decay of a discontinuity symmetric with respect to the bisectrix $\Gamma_1$ of the angle $\Gamma$ is later examined. By virtue of symmetry on $\Gamma_1$ the condition of impenetrability is satisfied. Let us introduce a fixed $x,y$ coordinate system in the flow plane so that at the time $t = 0$ the origin would coincide with the vertex of the angle and the $y$ axis would be directed along the side $\Gamma$. Let $\Gamma_1$ be given by the equation $y = -x \tan \alpha$ in this coordinate system. Then the initial data (1.1) should satisfy the relationships $v_1 = -u_1 \tan \alpha, v_0 = -u_0 \tan \alpha$. Without limiting the generality, we can consider that $u_0 = 0, p_1 \geq p_0$.

If new dependent and independent variables corresponding to the conical flows

$$
\xi = x / t, \quad \eta = y / t, \quad U = (U, V) = (u - \xi, v - \eta)
$$


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are introduced, then the system of gasdynamics equations will be reduced to
\[
(U \cdot \nabla) U + p \nabla p + U = 0
\]
\[
(U \cdot \nabla p) - \rho (\text{div} U + 2) = 0,
\]
\[
(U \cdot \nabla \delta) = 0
\]
(1.2)

This system is hyperbolic for \( |U|^2 > C^2 = \partial p/\partial \rho \) and elliptic for \( |U|^2 < C^2 \).

For large \( \eta \) the decay of the discontinuity is described by the known one-dimensional solution. Different configurations of the one-dimensional decay of a discontinuity are hence possible depending on the magnitudes of the constants prescribing the initial state (1.1). Following the terminology in the book [5], let us designate that decay of a one-dimensional discontinuity such that the shock goes into \( D_0 \) and the rarefaction wave into \( D_1 \) as the configuration A. The configuration B corresponds to two shocks going into \( D_0 \) and \( D_1 \), and the configuration C to two rarefaction waves.

The appropriate inequalities for the initial data (1.1) which will assure realization of the configurations mentioned are presented in [5]. Knowing the one-dimensional solutions, we can construct the boundary of the domain in which the flow will be essentially two-dimensional.

For large \( \eta \) let a one-dimensional decay of the discontinuity corresponding to configuration A occur (Fig. 1). The solution is constructed from the simple Riemann wave \( \psi_2 \) and the two constant flows adjoining the contact discontinuity. From the intersection of \( \Gamma_1 \) with the forward front of the simple wave let us draw a characteristic of the system (1.2) in the known solution \( \psi_2, 2 \) to intersect the front of the contact discontinuity at the point G. In the constant solution 3 let us construct a circle \( U^2 + V^2 = C^2 \) intersecting the front of the contact discontinuity at the point H. If \( \eta_H > \eta_C \) (as holds in Fig. 1), then a characteristic is drawn from the point H along the constant solution 2 to the intersection with the characteristic NG. The known boundary NMHF and the unknown shock front FE close the flow domain of a double-wave type. The boundary of the mentioned domain can be constructed analogously in the remaining cases also. A definite boundary-value problem for the system (1.2) with the unknown boundaries, the shock and contact discontinuity fronts, originates in the domain NMHF.

Let us consider the problem of the decay of a discontinuity in a linear formulation by assuming the angle \( \alpha \) small. Let us take the one-dimensional decay of a discontinuity for \( \alpha = 0 \) as the fundamental solution on which the linearization is carried out.

Configuration A. Some possible forms of the perturbed flow domain for the linear problem are represented in Figs. 2a, b, and c. The flow can be considered potential in the domain LMN. Let us introduce the flow potential by means of the formulas \( \psi_2 = U, \varphi_1 = V \) and let us represent the function \( \varphi \) as
\[
\varphi = \varphi_0 + a\psi
\]
where \( \varphi_0 \) is the potential corresponding to the fundamental solution. An equation for the perturbation potential \( \psi \) is obtained after linearization:
\[
\frac{1}{2} \left[ \eta^2 - \left( \frac{2C_1}{\gamma_1 + 1} - \frac{\gamma_1 - 1}{\gamma_1 + 1} (\xi - u_0) \right)^2 \psi_0 - \left[ \frac{2C_1}{\gamma_1 + 1} - \frac{\gamma_1 - 1}{\gamma_1 + 1} (\xi - u_0) \right] \times \right. \\
\times \eta \psi \right] - \frac{\gamma_1 - 1}{\gamma_1 + 1} \eta \psi_0 + \left[ \frac{2C_1}{\gamma_1 + 1} - \frac{\gamma_1 - 1}{\gamma_1 + 1} (\xi - u_0) \right] \psi_0 + \frac{\gamma_1 - 1}{\gamma_1 + 1} \psi = 0
\]
(1.3)

The boundary conditions for the function \( \psi \) are the following: \( \psi = -u_1 \eta \) on the characteristic LM, and \( \psi_\eta = -2(\gamma_1 + 1)^{-1}(C_1 + \xi) - (\gamma_1 - 1)(\gamma_1 + 1)^{-1}u_1 \) for \( \eta = 0 \). Linearizing the system (1.2) in the constant solutions 2, 3 results in the equations
\[
(xj, \nabla) u^j = \nabla p^j, \quad (xj, \nabla) p^j = \text{div} u^j, \quad (xj, \nabla) p^j = \text{div} u^j
\]
(1.4)

Here \( U^j, p^j, \rho^j (j = 2, 3) \) are the desired dimensionless perturbations defined as follows:
\[
u = u_j + aC_j u^j, \quad p = p_j + \rho_0 \rho^j \rho_j, \quad \rho = \rho_j (1 + \alpha\rho_j), \quad x_j = (\xi - U_0)/C_j, \quad y_j = \eta/C_j
\]
where \( p_j, \rho_j, C_j, U_j = (U_\theta, 0) \) are the gas parameters in the fundamental constant solutions. An equation for the function \( p^j \) follows from the system (1.4):
\[
(x_j^2 - 1) p^j_{xj} + 2x_j y_j p^j_{xj} + (y_j^2 - 1) p^j_{yj} + 2x_j p^j_{yj} + 2y_j p^j_{yj} = 0
\]
(1.5)