CONVECTIVE INSTABILITY OF A MIXTURE WITH CONCENTRATION HEAT SOURCES

A. K. Kolesnikov and V. I. Yakushin

We investigate the convective stability of a horizontal layer of a two-component binary mixture with internal heat release whose intensity depends on the concentration of one of the components. We present curves of neutral stability and graphs of the amplitudes of critical perturbations.

We consider a binary mixture, one of the components of which causes heat release. Internal sources of heat depending on the concentration ("concentration" sources of heat) can arise in a mixture as a result of processes of radioactive decay, selective absorption of light, or exothermic chemical reaction of nonzero order. An example of a mixture with concentration heat sources of radioactive type is the asthenospheric layer of the mantle of the earth [1]. Internal sources with intensity depending on the concentration also arise for the propagation of radiation in a layer with an impurity having large light absorption [2]. In this case the energy absorbed by an impurity can be converted into internal degrees of freedom, as a result of which there is rapid local heating near the impurity. Finally the model of a mixture with concentration heat sources gives a good description of certain types of exothermic chemical processes, operating with large thermal effects in a strongly diluted reagent.

Conditions of formation of convection in such system should be noticeably different from those for an ordinary nonisothermal binary mixture. The difference is connected first of all with the possibility of diffusion redistribution of heat sources.

We investigate the convective stability of an incompressible binary mixture with concentration heat sources. The mixture fills an infinite horizontal layer bounded by parallel isothermal planes $z = 0$ and $z = d$. On the lower boundary of the layer there is a constant concentration of heat-releasing component $C = C_0$; on the upper boundary $C = 0$. We assume that the density of the mixture depends linearly on the temperature and concentration

$$\rho = \rho_0 (1 - \beta_1 T - \beta_2 C),$$

where $\beta_1$ is the ordinary coefficient of thermal expansion, and $\beta_2 = -1/\rho_0 (\partial \rho/\partial C)_T$ determines the dependence of the density on concentration. For a light active component $\beta_2 > 0$; if the heat release is due to the heavy component, then $\beta_2 < 0$.

The system of equations describing the thermal-concentration convection in a binary incompressible mixture includes the equation of motion, the heat equation and diffusion equation, and the equation of continuity [3]. The presence of concentration sources of heat leads to the appearance in the heat equation of the additional term

$$\frac{Q}{\rho_0 c_p} C,$$

which depends linearly on the concentration of the active component, which corresponds, for example, to the exothermic reaction of first order. In expression (1), $Q$ is the specific intensity of heat release and $c_p$ is the specific heat.

With account of (1) the equations of convection in the binary mixture in dimensionless variables, assuming that the Boussinesq approximation is valid and that there is no thermal diffusion or diffusion heat conduction, take the form

$$\frac{\partial \mathbf{v}}{\partial t} + \frac{1}{P} (\mathbf{v} \mathbf{v}) \mathbf{v} = -\nabla P + \Delta \mathbf{v} + (RT + R_d C) \nabla T,$$

\[ P \frac{\partial T}{\partial t} + \nabla T = \Delta T + 6C, \]  
\[ P \frac{\partial C}{\partial t} + \frac{P_d}{P} \nabla C = \Delta C, \text{ div } \mathbf{v} = 0. \]  

In system (2) we use the ordinary notation (\( C \) is the dimensionless concentration of the heat-releasing component; the unit vector \( \gamma \) is directed vertically upward). As the units of measurement of distance, time, velocity, temperature, concentration, and pressure we choose the quantities \( d, d^2/\nu, \chi/d, qd^2, C(0), \text{ and } \rho_0 \nu \gamma/d^2 \), where \( q = QC(0)/6\rho_c p_0 \chi \).

The equations contain four dimensionless parameters: \( R = g\beta \chi qd^5/\nu \chi \) is the Rayleigh number; \( R_d = g\beta \chi (d^2/\nu) \), its concentration analog (the diffusion Rayleigh number); \( P = \nu/\chi \), the Prandtl number, and \( P_d = \nu/D \), the Schmidt number (the diffusion Prandtl number).

The boundaries of the layer are assumed to be rigid and are maintained at the same temperature assumed at the reference origin. On the lower boundary, as has already been noted, we are given a constant concentration \( C = 1 \); on the upper boundary the active component is absent. Thus, the velocity, temperature, and concentration satisfy the following boundary conditions:

\[ \begin{align*}
\text{for } z &= 0 \quad \mathbf{v} = 0, \quad T = 0, \quad C = 1; \\
\text{for } z &= 1 \quad \mathbf{v} = 0, \quad T = 0, \quad C = 0.
\end{align*} \]  

The boundary problem (2), (3) that has been formulated has a stationary solution corresponding to mechanical equilibrium:

\[ v_0 = 0, \quad T_0 = z(z^2 - 3z + 2), \quad C_0 = 1 - z. \]  

From the form of the equilibrium profiles of the temperature and concentration (4) it follows that in the layer there are regions with potentially unstable stratification of density, which is due to the temperature distribution, and in the case of a light active component this is also due to the distribution of concentration.

We investigate the stability of the distributions (4) with respect to the onset of convection. In order to do this we consider the behavior of the small normal perturbations \( \exp [-\lambda t + i(k_1 x + k_2 y)] \), where \( \lambda = \lambda_r + i\lambda_i; \lambda_r \) is the real part and \( \lambda_i \) is the imaginary part of the decrement \( \lambda \).

After linearization of the initial system (2) with respect to the small perturbations of velocity, temperature, and concentration, and elimination of the pressure, we obtain for their amplitudes \( w(z), \theta(z), \eta(z) \), a system of ordinary homogeneous differential equations

\[ \begin{align*}
-\lambda (w'' - k^2 w) &= (w^{1v} - 2k^2 w' + k^4 w) - Rk^2 \theta - R_d k^2 \eta, \\
-\lambda P\theta &= (\theta'' - k^2 \theta) + 6\eta - wT_0', \\
-\lambda P_d \eta &= (\eta'' - k^2 \eta) - \frac{P_d}{P} \frac{P}{T_0} wC_0'.
\end{align*} \]  

Here \( k^2 = k_1^2 + k_2^2 \).

The boundary conditions for \( w, \theta, \) and \( \eta \) in accordance with (3) have the form

\[ w = w' = \theta = \eta = 0 \text{ for } z = 0, 1. \]  

For \( P_d = P \) and \( R = 0 \) the problem (5), (6) is turned into a concentration analog of the known Rayleigh problem. For the case \( P_d = 0 \), Eqs. (5) with boundary conditions (6) describe the formation of convection in a layer with linearly distributed internal sources of heat [4]. For fixed concentration of the active component (\( C_0 = \text{const} \)) for \( P_d = 0 \), the problem (5), (6) reduces to the problem of the stability of a liquid with homogeneous heat release, which was considered in [5] for a thermally insulated lower boundary.

The decrements \( \lambda(P, P_d, R, R_d, k) \) are eigenvalues of the spectral problem (5), (6), and the amplitudes of the perturbations are its eigenfunctions.

For the solution, the system of equations for the complex amplitudes \( w, \theta, \) and \( \eta \) was reduced to a system of 16 real first-order equations for the real and imaginary parts of the functions \( w, w', w^", w^n, \theta, \theta^n, \eta, \) and \( \eta' \). The Runge-Kutta-Merson method [6] was used to construct four linearly independent particular