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Some Remarks on Maltsev and Goursat Categories

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Abstract. Our aim is to analyze and to publicize two interesting properties – well known in universal algebra for varieties – that a regular category, and in particular an exact category, may possess: the *Maltsev* property, asserting the permutability $SR = RS$ of equivalence relations on any object, and the weaker *Goursat* property, asserting only that $SRS = RSR$. We investigate these properties, give various equivalent forms of them, and develop some of their useful consequences.

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1. Introduction

In any finitely-complete category $\mathcal{A}$ we can define a *relation R from A to B* as a subobject of $A \times B$; but to define a composite $SR\colon A \rightarrow C$ of relations $R\colon A \rightarrow B$ and $S\colon B \rightarrow C$ we need $\mathcal{A}$ to admit *images*, and then this composition is associative precisely when these images are stable under pulling back. That is to say, the finitely-complete categories which admit an associative composition of relations are exactly the *regular* ones introduced by Barr in [1] and [2]. In such a category a relation $R\colon A \rightarrow A$ is of course said to be an *equivalence relation* if it is reflexive, symmetric, and transitive. For any map $f\colon A \rightarrow B$ in $\mathcal{A}$, the fibred product of f with itself, which we may write by analogy as $\{(a, b) \in A \times A \mid fa = fb\}$ and call the *kernel* of f, is an equivalence relation; those equivalence relations that arise thus as kernels are often said to be *effective*, but we shall rather call them *congruences*. By an *exact* category is meant a regular one in which every equivalence relation is a congruence; each *variety* (that is, the category of algebras for a one-sorted purely-equational algebraic theory) is an exact category. We devote Section 2 to a revision of the elementary properties of regular and exact categories, especially as concerns relations.

In any regular $\mathcal{A}$ the equivalence relations on an object A, ordered as subobjects of $A \times A$, form an ordered set Equiv A with finite meets $R \wedge S$ and sometimes – as for instance in a variety – also with finite joins $R \vee S$. The subset Cong A of Equiv A given by the congruences on A is closed under meets; moreover, being dual to the ordered set Quot A of quotient objects of A, it has finite joins if $\mathcal{A}$ admits

the appropriate pushouts; but these joins coincide with those in Equiv A *only* when $\mathcal{A}$ is exact – see Proposition 5.6 below, where we recall a result of Fay [10] to this effect. This article is largely concerned with the interplay between joins in Equiv A (rather than in Cong A, when these differ) and the operation of composition for relations. When the join $R \vee S$ in Equiv A exists, it certainly contains RS, SR, RSR, SRS, $RSRS$, and so on; in general these are all different – there are easy examples of this when $\mathcal{A}$ is the category of sets – and in any variety $R \vee S$ is their union; at the other extreme, however, it may be that $SR = RS$ for all R, S in Equiv A and for all A in $\mathcal{A}$ – in which case RS is clearly an equivalence relation and is hence the join $R \vee S$. By a classical result of Maltsev [25], this last holds in a variety $\mathcal{A}$ if and only if the theory of $\mathcal{A}$ contains a ternary operation m satisfying $mxxy = y$ and $mxyy = x$ – such as the operation in the theory of groups given by $mxyz = xy^{-1}z$. It is usual to call m a *Maltsev operation* and to call such an $\mathcal{A}$ a *Maltsev variety*; by extension, we define a *Maltsev category* to be a regular one having this permutability $SR = RS$ of equivalence relations – which we also call the *Maltsev condition*.

Our reason for doing so is that Maltsev categories – especially, but not exclusively, exact ones – have a number of very useful properties. For instance, Lambek pointed out in [24] that various diagram lemmas (including some non-additive generalizations of lemmas from classical homological algebra) hold, not only in the category of groups (where he had long ago investigated them in [23]), but more generally in any Maltsev variety; while more recently Carboni, Lambek and Pedicchio [5] observed that these lemmas hold equally well in *any* exact Maltsev category – it is a matter of finding proofs based not on the "syntactical" existence of a Maltsev operation, but on the "semantical" permutability $SR = RS$. We revise and refine these results below, and establish many further properties of Maltsev categories, some of which play essential roles in the forthcoming study by Janelidze and Kelly [17] of generalized central extensions.

Such a pursuit would have no value unless Maltsev categories were of fairly common occurrence in mathematics. In fact many familiar varieties are Maltsev – certainly, by the above, all of those in which part of the structure on an object is that of a group: for example the varieties of groups, abelian groups, modules over some fixed ring, rings, commutative rings, associative algebras, Lie algebras, and so on. The variety of Heyting algebras provides (see [18], p. 9) an example not of this kind; whence the variety of boolean algebras gives an example. Of course there are exact Maltsev categories that are not varieties; for the product $\mathcal{A} \times \mathcal{B}$ of exact Maltsev categories is exact Maltsev, as is the functor category $[\mathcal{K}, \mathcal{A}]$ when $\mathcal{A}$ is so, along with each slice-category $\mathcal{A}/A$ and each slice category $A/\mathcal{A}$. Moreover, every abelian category is exact Maltsev; for abelian categories are exact – Tierney (see [2], p. 12) showed them to be precisely the exact additive categories – while (see Johnstone [19] or Proposition 3.7 below) it is easy to see that *every additive regular category is Maltsev.* Among the abelian examples of exact Maltsev categories that are not varieties are the duals of the categories of