Earth’s Rotational and Orbital Motions Interrelated by Loitsyanski’s Turbulence Theorem

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Summary – The Loitsyanski theorem for turbulent fluid motion is presented in simplified form from the viewpoint of angular momentum conservation and extended by a normalization with respect to total mass. When the new parameter thus obtained is evaluated for a gas cloud of the size and mass of the present Sun-Earth system, and this is equated to the corresponding parameter for the Earth-Moon configuration, an accurate equation between independently determined astronomical constants is found. The Earth’s rotational speed is expressed in terms of its mean orbital motion, the Sun-Earth mass ratio, the ratio of separation distances, and the Earth’s radius. The most simple form of the algebraic relationship (10) is believed unique in linking planetary orbital dynamics to motion about the center of gravity.

1. Modification of Loitsyanski’s Theorem

The total angular momentum of a uniform fluid is given by the integral of particle position vector product with the velocity vector:

$$ H = \rho \int \mathbf{r} \times \mathbf{q} \, dV. $$

Integration extends over the whole volume, and \( \rho \) denotes constant fluid density. A transformation of this expression is required which is obtained most easily by considering a single component of the angular momentum, written in scalar form as

$$ h = \rho \int (x \, u - y \, v) \, dV, $$

where \( u \) and \( v \) denote the \( x\)- and \( y\)-components of velocity. Equation (2) gives the component of total angular momentum (1) perpendicular to the plane of \( x \) and \( y \). Considering a bounded region of fluid, the normal component of its velocity vanishes on the boundary, suggesting that the first term in (2) be written as

$$ \int x \, v \, dV = \int \nabla \cdot (x \, y \, \mathbf{q}) \, dV - \int y \, u \, dV. $$

The first term on the right side of (3) is equal to the surface integral of the normal velocity component, hence it vanishes. In obtaining (3) use has been made of the

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assumption of uniformity of the fluid, so that the mass conservation condition reduces to $\nabla \cdot q = 0$. Thus (2) becomes

$$h = -2 \frac{g}{\rho} \int y u \, dV$$

and the square of this quantity can be expressed as a repeated integral

$$h^2 = 4 \frac{g^2}{\rho^2} \int y u \, dV \int y' u' \, dV' = -2 \frac{g^2}{\rho^2} \iint (y - y')^2 u u' \, dV \, dV'$$

because the total linear momentum of the fluid, which can be written as

$$\int u \, dV$$

is zero in a center-of-mass coordinate system. The last expression in (4) is interpreted as referring to the velocity components $u$ and $u'$ at distinct points $y$ and $y'$ in regions $dV$ and $dV'$ respectively.

The squared angular momentum is evaluated as

$$h^2 = 2 \frac{g^2}{\rho^2} \int f \, dV,$$

where

$$f = -\int (y - y')^2 \overline{u u'} \, dV$$

and the bar indicates an appropriate time-averaging of velocities $u$ and $u'$ at neighboring points of fluid. The angular momentum (2) is then no longer an instantaneous value but a mean value, denoted here by the same symbol. Correlation functions in the statistical theory of turbulence are defined with the aid of such quantities as (6), and their properties are only partially known. It is considered that the integrand in (6) falls off rapidly with increasing values of $r^2 = (y - y')^2$ since the velocities of a turbulent flow at two widely separated points are regarded as statistically independent. In isotropic flow the quantity $f$ is a function of $r$ only, and the squared angular momentum (5) is written as

$$h^2 = 4 \frac{g^2}{\rho^2} f V$$

when the motion is assumed homogeneous except for a very small part near the boundary.

The constancy of angular momentum $h$ implies that $f$ is constant in isotropic turbulent flow. This is the substance of LIOPTYSANSKII’s theorem, when $f$ is understood as an integral of $r^2 \overline{q \cdot q}$ in three-dimensional motion. If the integrand of (6) is appreciable in a region of dimension of the order of the scale $l$ of the turbulence, then the volume is of order $l^3$ and it is seen from (6) that

$$u^2 l^5 = \text{constant}. \quad (8)$$

The angular momentum in (5) is proportional to the total mass $M$ of fluid and to its speed $q$ so that $h \propto M \cdot l \cdot q$ and thus

$$h^2 \propto M^2 q^2.$$