The 'Missing' Member of Hough's Functions in Diurnal Tides

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Summary – It is shown that for azimuthal wave number \( s \) and period \( 0.5 (s + 1) \) days, Hough's functions for tidal oscillations are the same as the associated Legendre polynomials \( P_{s+1} \) and Hough's functions are shown to form a complete set of orthogonal functions. The implication of this on the vertical variation of tides is discussed.

1. Introduction

The equation governing atmospheric tidal motion for a particular frequency and for a particular longitudinal wave number (see for example Siebert [1]) is of the form

\[
Z(\Psi) + \frac{g}{4a^2 \omega^2} F \left( q - \left( \frac{dH}{dZ} + K \right) \Psi \right) = 0, \tag{1}
\]

where the operators \( Z \) and \( F \) are given by

\[
Z(\Psi) = \frac{H \partial^2 \Psi}{\partial Z^2} + \left( \frac{dH}{dZ} - 1 \right) \frac{\partial \Psi}{\partial Z} \tag{2}
\]

\[
F(\Psi) = \frac{1 - x^2}{f^2 - x^2} \frac{\partial^2 \Psi}{\partial x^2} + \frac{2(1 - f^2) x \partial \Psi}{(f^2 - x^2)^2 \partial x} - \left( \frac{1}{f^2 - x^2} \right) \left( \frac{1}{f} \left( \frac{f^2 + x^2}{f^2 - x^2} \right) + \frac{s^2}{1 - x^2} \right) \Psi \tag{3}
\]

and

\( H \) = pressure scale height of the atmosphere;
\( K = 2/7 \), i.e. \( (\gamma - 1)/\gamma \);
\( \Psi = \chi - \frac{KJ}{gH} \), and \( \chi \) is the velocity divergence;
\( q = KJ/(1.4gH) \) and \( J \) = heat energy absorbed per unit mass per second;
\( g \) = acceleration due to gravity;
\( a \) = radius of the earth;
\( \omega \) = angular velocity of rotation of the earth round its axis;

\( f = \sigma/2 \) and \( i \sigma = \frac{\partial}{\partial t} \).

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This is an inhomogeneous equation with \( q \) as the forcing function. The usual method of solving it is by the separation of variables, that is we assume that

\[
q = \sum_n q_n^s(z) \theta_n^s(x) \quad \text{and} \quad \Psi = \sum_n \phi_n^s(z) \theta_n^s(x),
\]

where the \( \theta_n^s \) are the eigenfunctions called Hough's functions, obtained by solving the equation

\[
F\{\theta_n^s(x)\} + \lambda_n^s \theta_n^s(x) = 0,
\]

so that

\[
Z\{\phi_n^s(z)\} + \left( \frac{dH}{dZ} + K \right) \phi_n^s(z) q_n^s(z) \frac{1}{h_n^s} = 0
\]

and the \( \lambda_n^s \) are the eigenvalues of the operator \( F \) and

\[
h_n^s = 4a^2 a^2/(\lambda_n^s g).
\]

It is worthwhile pointing out that our use of suffixes \( n \) and \( s \) in \( \theta_n^s \) is in keeping with the standard use of the suffixes in Legendre associated functions; that is \( n - s \) is the number of nodes between the poles and not including the poles.

2. The completeness of \( \theta_n^s \)

The procedure for solving equation (1) outlined above is correct provided the Hough functions form a complete set of orthogonal functions; then an arbitrary form of heating function at any level can be obtained by adding up the various \( \theta_n^s \) with the appropriate factor \( q_n^s(z) \).

However, Hough [5] pointed out that for the case \( f = 1/(s + 1) \), 'all' the Hough functions are orthogonal to Legendre associated function of order \( s + 1 \), i.e. \( P_{s+1}^s(x) \). On account of this Lindzen [2], felt that the Hough functions for this special case in question do not form a complete set, since a necessary condition for a set to be completed is that there shall not be any other function which is orthogonal to all the members of the set.

The purpose of this note is to show that, contrary to this view, Hough's functions form a complete set. To this end, in the next section we shall prove that \( \theta_{s+1}^s \) and \( P_{s+1}^s \) are one and the same function.

3. Mathematical proof of \( \theta_{s+1}^s = P_{s+1}^s \) for \( f = 1/(s + 1) \)

It can easily be shown that the Legendre associated function \( P_n^s(x) \) satisfies

\[
x(1-x^2) \frac{d}{dx} P_n^s + \left( n^2 - \frac{(n^2 - s^2)}{2n - 1} \right) P_n^s = \frac{(n + s)(n + s - 1)}{(2n - 1)} P_{n-2}^s.
\]