Ideals, Radicals, and Structure of Additive Categories

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Abstract. Simple and semisimple additive categories are studied. We prove, for example, that an artinian additive category is (semi)simple iff it is Morita equivalent to a division ring(oid). Semiprimitive additive categories (that is, those with zero radical) are those which admit a noether full, faithful functor into a category of modules over a division ringoid.


Key words: Additive category, semiprimitive ring, artinian, simple, Jacobson, radical, density arguments.

The idea that additive categories are rings with several objects was developed convincingly by Barry Mitchell [8] who showed that it is unusual for a theorem of (non-commutative) ring theory not to carry over to additive categories. Here we would like to further argue that insight and efficiency (in concepts, statements, and proofs) are to be gained by dealing with additive categories throughout, and that familiar theorems for rings come out of the natural development of category theory. In other words, we attempt to apply additive category theory to ring theory rather than to generalize ring theory to additive categories. As a typical advantage to this approach we point to the fact that a ring and its category of finitely generated projective modules can be treated on an equal footing.

The radical of an additive category was defined by G. M. Kelly [7]. One of our purposes is to analyse this radical in more detail and to investigate its relation to a notion of semisimple additive category. An additive category is called semiprimitive when its radical is zero, and we provide a characterization of these categories in terms of a categorical concept called noether fullness. We were inspired by a preprint of Karlheinz Baumgartner [2], and, while we claim little in the present paper is really new, the results seem largely unknown and without a uniformly categorical published treatment.

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Let $\text{Ab}$ denote the category of (small) abelian groups. All categories and functors will be additive (meaning $\text{Ab}$-enriched) without further mention. So a category with only one object amounts to a ring (with identity).
For any category $\mathcal{A}$, a (right) $\mathcal{A}$-module is a functor $M : \mathcal{A}^{\text{op}} \to \text{Ab}$; if $f \in \mathcal{A}(A, B)$ and $m \in M(B)$, we write $mf$ for $M(f)(m) \in M(A)$. We write $\text{Mod}\mathcal{A}$ for the category of $\mathcal{A}$-modules and natural transformations between them. There is a fully functor $\mathcal{Y}_\mathcal{A} : \mathcal{A} \to \text{Mod}\mathcal{A}$, called the Yoneda embedding, which takes each object $X$ of $\mathcal{A}$ to the representable module $\mathcal{Y}_\mathcal{A}(X) = \mathcal{A}_X$ where $\mathcal{A}_X(A) = \mathcal{A}(A, X)$. Let $\mathcal{Q}\mathcal{A}$ denote the full subcategory of $\text{Mod}\mathcal{A}$ consisting of those modules which are retracts of finite direct sums of representable modules $\mathcal{A}_X$, $X \in \mathcal{A}$. A module $M$ is in $\mathcal{Q}\mathcal{A}$ iff the representable functor $(\text{Mod}\mathcal{A})(M, -) : \text{Mod}\mathcal{A} \to \text{Ab}$ preserves colimits. The category $\mathcal{Q}\mathcal{A}$ is called the projective (or Cauchy) completion of $\mathcal{A}$. If $\mathcal{A}$ is a ring, $\mathcal{Q}\mathcal{A}$ is the category of finitely generated, projective $\mathcal{A}$-modules. A category $\mathcal{A}$ is called projectively complete when it has finite direct sums and splittings for all idempotents; this holds iff $\mathcal{Y}_\mathcal{A} : \mathcal{A} \to \mathcal{Q}\mathcal{A}$ is an equivalence of categories. Clearly $\mathcal{Q}\mathcal{A}$ is projectively complete for all $\mathcal{A}$. Categories $\mathcal{A}, \mathcal{B}$ are called Morita equivalent when $\text{Mod}\mathcal{A}$ is equivalent to $\text{Mod}\mathcal{B}$. A basic Morita-style theorem (true very generally for enriched categories; see [9] for example) is that $\mathcal{A}, \mathcal{B}$ are Morita equivalent iff $\mathcal{Q}\mathcal{A}$ is equivalent to $\mathcal{Q}\mathcal{B}$. For any $\mathcal{A}$, clearly $\mathcal{A}$ and $\mathcal{Q}\mathcal{A}$ are Morita equivalent.

An object of a category $X$ is called artinian (respectively, noetherian) when every descending (respectively, ascending) chain of subobjects is finite. Call $\mathcal{A}$ artinian when each $\mathcal{A}_B$ is an artinian object of $\text{Mod}\mathcal{A}$. In this case, each object of $\mathcal{Q}\mathcal{A}$ is an artinian $\mathcal{A}$-module, and $\mathcal{Q}\mathcal{A}$ is artinian.

For additive categories $\mathcal{A}, \mathcal{B}$, the tensor product $\mathcal{A} \otimes \mathcal{B}$ is the additive category whose objects are pairs $(A, B)$ of objects $A \in \mathcal{A}$, $B \in \mathcal{B}$, and whose homs are given by

$$(\mathcal{A} \otimes \mathcal{B})((A, B), (A', B')) = \mathcal{A}(A, A') \otimes \mathcal{B}(B, B').$$

For each $\mathcal{A}$, there is a hom functor $\mathcal{H}_\mathcal{A} : \mathcal{A}^{\text{op}} \otimes \mathcal{A} \to \text{Ab}$ given by $\mathcal{H}_\mathcal{A}(A, B) = \mathcal{A}(A, B)$. In other words, $\mathcal{H}_\mathcal{A}$ is an $(\mathcal{A} \otimes \mathcal{A}^{\text{op}})$-module.

Before discussing the radical, we make some remarks about general ideals. Given an object $X \in \mathcal{A}$, a right $X$-ideal $R$ of $\mathcal{A}$ is a submodule of $\mathcal{A}_X \in \text{Mod}\mathcal{A}$. It can be regarded as a set $R$ of arrows into $X$ such that, if $f, g : A \to X$ are in $R$ and $v : C \to A$, then $(f + g)v$ is in $R$. This agrees with the definition of right ideal when $\mathcal{A}$ is a ring. Note that $R = \mathcal{A}_X$ iff $R$ contains a retraction.

An $\mathcal{A}$-module $M$ is called simple (or "irreducible") when it has precisely two distinct submodules $0 \subseteq M$ and $M \subseteq M$. An $\mathcal{A}$-module $M$ is called semisimple (or "completely reducible") when it is a direct sum of simple modules. Recall ([3] Ch. 1, Proposition 4.1) that a module is semisimple iff each submodule is a direct summand (the proof there works for $\mathcal{A}$-modules without change).

An ideal $\mathcal{K}$ in a category $\mathcal{A}$ is a submodule of $\mathcal{H}_\mathcal{A} \in \text{Mod}(\mathcal{A}^{\text{op}} \otimes \mathcal{A})$. We can identify an ideal $\mathcal{K}$ with the union of all the sets $\mathcal{K}(A, B)$ of arrows in $\mathcal{A}$; a set $\mathcal{K}$ of arrows in $\mathcal{A}$ is an ideal iff, for all $f, g : A \to B$ in $\mathcal{K}$, the arrow $u(f + g)v : X \to Y$ is in $\mathcal{K}$ for all $u : B \to Y$, $v : X \to A$. We recapture the submodule via $\mathcal{K}(A, B) = \mathcal{A}(A, B) \cap \mathcal{K}$. Each $\mathcal{K}(A, A)$ is an ideal of the