STATE OF STRESS IN A CIRCULAR RING INSERTED
IN A CIRCULAR HOLE OF AN EXTENDED PLATE

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1. We consider the problem about the state of stress of an elastic isotropic ring S2 with inner radius
R0 and outer radius R2, inserted in a circular hole of an unbounded isotropic medium (plate) S1 whose radius
is R = R1 (Fig. 1). We shall consider two cases of fitting the ring in the hole: 1) the ring is inserted with zero
tolerance (R1 = R2); 2) the ring is embedded with an initial clearance Δ = R2−R1. The stresses in the ring
and in the plate appear under the action of a system of loads p and q, applied to the plate in two mutually
perpendicular directions.

In the first case, under the action of the loads p and q, the force contact between the bodies S1 and S2
appears at separate parts L1 and L2 of their contours. On the arcs L1 and L2 there will appear only normal
(radial) stresses σr (we neglect the friction between the ring and the plate). In the second case, the contact
between the bodies S1 and S2 occurs along the entire length of their contact contours.

We introduce a system of rectangular Cartesian coordinates xOy, as shown on Fig. 1. Then the state
of stress of the plate S1 at infinity is given by

σr(∞) = p, \quad \tau_{r\theta}(∞) = 0, \quad \sigma_\theta(∞) = q.

In accordance with the formulation of the problem, on the contour of the circular hole and on the outer
contour L of the ring we have the following boundary conditions:

\begin{align}
\sigma_r(\alpha) = 0 & \quad \text{on } L = L_1 - L_2; \\
\tau_{\alpha\alpha}(\alpha) = 0 & \quad \text{on } L; \\
k_1(\alpha) = k_2(\alpha) & \quad \text{on } L_1 \text{ and } L_2,
\end{align}

where σr and τ_{\alpha\alpha} are the normal and tangential stresses; k1(\alpha) and k2(\alpha) are the curvature of the boundaries
of the bodies S1 and S2, deformed as a result of the contact of these bodies; \alpha is the polar angle measured from
the Ox axis in counterclockwise direction (see Fig. 1).

Making use of the known results of [2, 4, 6], we have found the following integrodifferential equation
for the contact pressure on the setting surface between the ring and the plate:

\begin{align}
\frac{1}{\pi} \int_{-\pi}^{\pi} \operatorname{ctg}(\alpha - \theta) \rho'(\theta) \, d\theta + \gamma_1 \rho(\alpha) - \frac{1}{\pi} \gamma_2 (\gamma_3 - 2\gamma_2 \rho_0) C_\alpha \\
+ 2\gamma_3 \left[ \frac{3}{2} (p - q) \cos 2\alpha - \frac{1}{4} (p + q) \right] - \frac{2}{\pi} \gamma_3 \sum_{n=1}^{\infty} [d_{2n} (2n - 1)]
\end{align}


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Here $p(\alpha) = -\sigma_T(\alpha)$ is the contact pressure; $-\alpha_0 \leq \alpha \leq \alpha_0$; $\alpha_0$ is the angle of contact; $\rho'(\alpha) = d\rho(\alpha)/d\alpha$; $\gamma_1 = \left(\gamma_1 - 1\right) G_1/(\gamma_1 - 1) G_1/h$; $\gamma_2 = \left(\gamma_2 + 1\right) G_2/h$; $\gamma_3 = \left(\gamma_3 + 1\right) G_3/h$; $\gamma_4 = \left(\gamma_4 + 1\right) G_4/h$; $\alpha_0 = 3 - 4\nu_1$ for the plane strain, $\nu_1 = (3-4\nu_1)/(1+\nu_1)$ for the generalized state of plane stress; $\nu_1$ is the Poisson ratio, $G_1$ is the shear modulus ($i = 1$ and 2 for the plate and the ring, respectively);

$$C_0 = \frac{\alpha_0^2}{2} \int_{-\alpha_0}^{\alpha_0} \rho(\alpha) d\alpha; \quad C_n = \frac{\alpha_n^2}{2} \int_{-\alpha_n}^{\alpha_n} \rho(\alpha) \cos 2n\alpha d\alpha; \quad \alpha_n = \left(\frac{n}{2}\right) \left(1 - 2\nu_1\right) \left(1 + \nu_1\right);$$

$$a_{2n} = \left(1 - 2n\right) \lambda^{2n} + 2n\lambda^{2n-2}; \quad A_{-2n} = \lambda^{2n} \quad (n = 1, 2, \ldots);$$

$$\alpha_0 = \frac{\lambda^2}{2 \left(1 - \lambda^2\right)}; \quad \lambda = \frac{R_0}{R_2}.$$ (1.6)

The coefficient $a_{-2n}$ is obtained from the formula (1.6) by changing the sign of $n$.

For $\lambda = 0$ and $R_1 = R_2$ from (1.4) we obtain the equation for the case when a solid disk is inserted in the hole of the plate [4, 7, 8].

2. We consider the equation (1.4) and we find its approximate solution for $\Delta = 0$. To this end, we perform in (1.4) the change of variables $\tan \phi = a \cos \phi$; $\tan \alpha = \cos \theta$; $\alpha = \tan \alpha_0$. As a result of this substitution, equation (1.4) and the relations (1.5) become:

$$\frac{1 + a^2 \cos^2 \theta}{na} \int_{0}^{\pi} \frac{\rho'(\phi) d\phi}{\cos \phi - \cos \theta} + \gamma_2 \rho(\theta) \frac{\gamma_2 + \gamma_n^2}{\pi} \left(\lambda^2 + \frac{1}{1 - \lambda^2}\right) C_\theta$$

$$= \frac{\gamma_2}{\pi} \sum_{n=1}^{\infty} \left[ a_{2n} (2n - 1) a_{-2n} (2n + 1) \right] C_n \cos [2n \arctan (a \cos \theta)]$$

$$= \gamma_2 \left[ \frac{1}{2} \left(1 + \frac{p}{q}\right) + 3 \left(1 - \frac{p}{q}\right) \frac{1 - a^2 \cos^2 \theta}{1 + a^2 \cos^2 \theta} \right] \left(0 \leq \theta \leq \pi\right);$$ (2.1)

$$C_0 = a \int_{\theta}^{\pi} \rho(\theta) \sin \theta d\theta; \quad C_n = a \int_{\theta}^{\pi} \frac{\rho(\theta) \cos [2n \arctan (a \cos \theta)] \sin \theta d\theta}{1 + a^2 \cos^2 \theta}. \quad (2.2)$$

We represent the desired contact pressure in the form of a Lagrange trigonometric polynomial [1]:

$$L[\rho; \xi] = \frac{2}{N + 1} \sum_{n=1}^{N} \rho(\xi_n) \sum_{m=1}^{N} \sin m\theta_n \sin m\theta; \quad \xi = \cos \theta.$$ (2.3)