ON THE PROPAGATION OF VISCO-ELASTIC WAVES OF FINITE AMPLITUDE

V. G. Karnaukhov

§1. The fundamental equations of the nonlinear theory of elasticity and visco-elasticity in the Lagrangian and Eulerian representations in an arbitrary curvilinear system of coordinates is given in [2, 4, 9]. To within quantities of the second order of smallness inclusive these equations, written in cartesian Eulerian coordinates, have the form

\[ \sigma_{i j, k} = \rho (\ddot{\sigma} + v_i \dot{\sigma}_k); \]  
\[ \dot{\rho} = \rho_0 (1 - \varepsilon); \]  
\[ \dot{v}_i = \dot{u}_i + \dot{u}_j \dot{u}_{i,j}; \]  
\[ \sigma_{i j} = \lambda \delta_{i j} + 2\mu \varepsilon_{i j} + (\lambda + 2\mu) \varepsilon_{i j}; \]  
\[ \varepsilon_{i j} = \frac{2}{\lambda + 2\mu} \left( \frac{\varepsilon_{i k} - \varepsilon_{k i}}{2} \right). \]

(1.1)

(1.2)

(1.3)

where

\[ \lambda = \frac{\mu}{\lambda + 2\mu}; \]

\[ \mu = \frac{\lambda + 2\mu}{2}. \]

Here \( I_1 \) and \( I_2 \) are the first and second invariants of the deformation tensor; (*) indicates the operation of convolution. The remaining notation is that commonly used.

The boundary conditions have the form

\[ \sigma_{i j} n_j = p_i. \]

where \( n_j \) is the normal to the deformed surface of the body.

Since solution of these equations involves great mathematical difficulties, it becomes necessary to simplify the initial system of equations at the expense of restricting the class of problems. Significant mathematical simplifications can be achieved for problems involving the propagation of waves in rods, plates, and shells; in addition, to take into account the effects of geometric dispersion use can be made of refined theories of a different sort, say, of Timoshenko type, and to take into account dispersion of a physical nature and of dissipation use can be made of the theory of visco-elasticity.

For a study of the basic peculiarities of the interaction of the effects of nonlinearity, dispersion, and dissipation we examine the propagation of waves in visco-elastic rods of arbitrary cross section. We use the following method for deriving solutions of the equations; this method can also be used in a number of other problems. Assuming that the effects of dispersion, dissipation, and nonlinearity are small quantities of the same order, we can, in deriving the equations, take into account at first only the nonlinearity and visco-elasticity and then consider the terms creating geometric dispersion as being known from more refined linear rod theories.

In accordance with the nature of the motion under study we put

\[ \sigma_{i j} = \sigma_{23} = \sigma_{12} = \sigma_{13} = 0; \quad \sigma_{33} \neq 0. \]

(1.4)
Then from the equations of motion we have

\[ u_{1,2} = u_{1,3} = u_{2,1} = u_{2,3} = u_{3,1} = u_{3,2} = 0. \]

Putting \( u_3 = F(x) \), we obtain a system of nonlinear integral equations for determining \( u_{1,1} \) and \( u_{2,2} \) from the conditions (1.4) and the equations of state (1.3). Solving this system by the method of successive approximations, we obtain

\[ \sigma_{33} = EF_{33} + (2 \nabla_0 - c_0) F_{33} + (2 \nabla_d - d_0) F_{33}; \]

\[ c_0 = \frac{1}{2} \lambda (1 + 2v) - \lambda_1 (1 - 2v)^2 + iv (2 - v) + \mu v^2 + 2m (1 - 2v) - 4p; \]

\[ d_0 = -(1 - 2v)\lambda + 2\nabla; \quad \ddot{d}_0 = - [(1 - 2v)\lambda + 2\nabla]. \]  

(1.5)

We substitute equations (1.5) into the equation of motion (1.1). Taking into account the equations (1.2) and (1.4) and introducing the Love linear correction to the geometric dispersion [7] and also the dimensionless quantities

\[ \sigma = \frac{\sigma_m}{E}; \quad \Phi = \frac{F}{h}; \quad \tau = \frac{t}{h^2} \left( \frac{E}{\rho_0} \right)^{\frac{1}{2}}; \quad z = \frac{x_0}{h} \]

(\( R \) is a characteristic linear dimension), we obtain, to within second order quantities of smallness inclusive, the following integrodifferential equation:

\[ \beta_1 \frac{\partial^3 \Phi}{\partial z^3} + \beta_2 \frac{\partial^3 \Phi}{\partial z^2} + \beta_3 \frac{\partial^3 \Phi}{\partial z} + \beta_4 \frac{\partial^3 \Phi}{\partial z^2} + \frac{\partial^3 \Phi}{\partial z^3} = \frac{\partial^3 \Phi}{\partial z^3} + 2 \frac{\partial^3 \Phi}{\partial z^2} + 2 \frac{\partial^3 \Phi}{\partial z} + \beta_5 \frac{\partial^3 \Phi}{\partial z^2}; \]

(1.6)

where

\[ \beta_1 = \frac{1}{2} \nu^2; \quad \beta_2 = \frac{1}{E} \left( 4 \nu c_0 - 2c_0 \right); \quad \beta_3 = \frac{1}{E} \left( 2 \nu d_0 - d_0 \right). \]

To within quantities of the second order of smallness we write the equation (1.6) in the form of the system

\[ \frac{\partial v}{\partial \tau} + v \frac{\partial v}{\partial z} + \frac{\partial \left( H \right)}{\partial z} = 1 - \frac{\partial \Phi}{\partial z}; \quad c^2 = 1 + 2 \left( \frac{\beta_3}{2} - v \right) - 2 \left( \frac{\beta_3}{2} - v \right) H. \]  

Here

For \( \beta_3 = 0 \) these equations are known as the Boussinesq equations (see [6]). If we neglect the effects of dispersion and dissipation, these equations are of the same form as the gasdynamic equations, so that all the results valid for these latter equations become applicable (Riemann invariants, simple waves, etc). Moreover the nonlinear effects lead to a distortion of the wave profile, as a result of which the function \( v(z, \tau) \) becomes ambiguous for sufficiently large \( \tau \). However even before this time the solution becomes unsuitable since when the wave profile becomes sufficiently steep the dissipative processes become important, leading to a spreading out of the wave profile and, as a final result, a balancing of the nonlinear distortion. Dispersion also leads to a smearing out of the profile, which may compensate the nonlinear distortion. Therefore in dissipative and dispersive media the propagation of nonlinear stationary waves is possible, although their structures in each of these media are not identical [6]. Moreover in such media, for sufficiently small (but finite) amplitudes, solutions exist which can be considered as the analog of simple waves, namely, the so-called "quasi-simple waves." In order to obtain the equations for "quasi-simple waves" we use the method given in [6], wherein in simplicity we put \( \lambda = \lambda_0 \frac{\partial}{\partial \tau}; \mu = \mu_0 \frac{\partial}{\partial \tau}; \beta_3 = \beta_3 \frac{\partial}{\partial \tau}; \phi_3 = \frac{1}{E} \left( 2 \nu d_0 - d_0 \right); \phi_0 = \text{const}. \) Changing to a fixed system of coordinates \( x = \tau - \tau \), with the aid of this method we obtain, from the equation (1.6) and also the system (1.7), the Korteweg–de Vries–Burgers (KVB) equation

\[ f_t + f_x + \beta f_{xxx} = \mu f_{xx}. \]

(1.8)