ON THE REISSNER–SAGOCI PROBLEM

M. I. Chebakov

UDC 539.3;534.1

The problem of the torsional oscillations of an elastic halfspace caused by the rotation of a circular punch has been studied for the first time by E. Reissner and H. F. Sagoci [8]. They have obtained the exact solution of the problem, valid for low oscillation frequencies of the punch. The papers [4, 5, 6, 7, 9] are also devoted to the investigation of the low oscillation frequencies of the punch. For high frequencies one has obtained solutions in [10, 11].

Below we give three approximate methods for the solving of the formulated problem which generalize the methods of [1], have a simple structure and are efficient in practical use. Two of them give the asymptotic solution for small frequencies while the third one for large frequencies.

1. Formulation of the Problem. Assume that a circular punch of radius \( a \), linked to a halfspace, is subject to the torsional moment

\[ M = M_0 e^{i \omega t}. \]  

By the methods of operational calculus, this problem can be reduced to the solving of the following integral equation of the second kind (see, for example, [5]):

\[ \Phi(t) = \int_1^\infty \Phi(\tau) k\left( \frac{T - \tau}{\lambda} \right) d\tau + t \quad (|t| < 1); \tag{1.2} \]

\[ k(y) = \int_0^\infty [1 - L(u)] \cos uydu = \frac{\pi}{2} \left[ J_1(|y|) - iH_1(y) + \frac{2i}{\pi} \right]; \tag{1.3} \]

\[ L(u) = \frac{u}{\sqrt{u^2 - 1}}. \tag{1.4} \]

Here \( 1/\lambda = a \omega \rho^{(1/2)} G^{-(1/2)} \); \( G \) is the shear modulus, \( \rho \) is the density, \( J_1(y) \) is the Bessel function, \( H_1(y) \) is the Struve function; equation (1.2) is written in dimensionless variables.

The contact stresses \( \tau_{z \varphi} = \tau(r) e^{i \omega t} \) under the punch and the moment (1.1) are expressed in terms of the solution of the equation (1.2) with the formulas

\[ \tau(r) = -\frac{4Ge}{\pi} \int_0^r \frac{\Phi(\tau)}{\sqrt{\tau^2 - x^2}} d\tau \quad (x = \frac{r}{a}); \tag{1.5} \]

\[ M_0 = 16Gea^3 \int_0^\frac{1}{2} \tau \Phi(\tau) d\tau \tag{1.6} \]

(\( \varepsilon \) is the complex amplitude of the rotation angle of the punch).

2. The Method of Large Values of \( \lambda \). For sufficiently large values of the parameter \( \lambda \) we will seek the solution of the equation (1.2) in the form [1]

\[ \Phi(t) = \sum_{k=0}^\infty \Phi_k(t) \lambda^{-k}. \tag{2.1} \]

Making use in the kernel (1.3) of the representations of the Bessel and Struve functions, substituting (2.1) into (1.2), and making equal the coefficients of the same powers of \( \lambda \), we obtain recursive relations for

the determination of the coefficients \( \Phi_k(t) \) in (2.1)

\[
\Phi_0(t) = t; \quad \Phi_1(t) = 0;
\]

\[
\Phi_n(t) = \frac{1}{\pi} \sum_{m=0}^{n-2} b_{n-m-1} \int_1^{\infty} \Phi_m(\tau) \tau^{n-m-1} d\tau \quad (n \geq 2),
\]

where

\[
b_{2n} = \frac{n!2^n(-1)^n}{(2n+1)!!(2n)!}; \quad b_{2n+1} = \frac{(-1)^n\pi}{n!(n+1)12^{2n+2}} \quad (n = 1, 2, 3, \ldots).
\]

Restricting ourselves to terms of order \( \lambda^{-1} \) in (2.1), taking into account (2.2), we obtain

\[
\Phi(t) = \sum_{k=0}^{7} \Phi_k(t) \lambda^{-k} + O(\lambda^{-8}); \quad \Phi_0(t) = t;
\]

\[
\Phi_1(t) = 0; \quad \Phi_2(t) = a_1 t^2 + a_2 t; \quad \Phi_3(t) = ia_3 t;
\]

\[
\Phi_4(t) = a_3 t^4 + a_6 t^4 + a_9 t; \quad \Phi_5(t) = i(a_3 t^4 + a_6 t); \\
\Phi_6(t) = a_6 t^6 + a_{10} t^6 + a_{13} t^6 + a_{16} t; \\
\Phi_7(t) = i(a_{13} t^6 + a_{16} t^6 + a_{19} t).
\]

Here

\[
a_1 = 0.083333; \quad a_2 = -0.25000; \quad a_3 = 0.14147;
\]

\[
a_4 = -0.0033187; \quad a_6 = 0.010417; \quad a_9 = 0.098958;
\]

\[
a_{10} = -0.0070736; \quad a_{11} = -0.0047743; \quad a_{12} = -0.0017036;
\]

\[
a_{13} = 0.0018947; \quad a_{14} = 0.0035620; \quad a_{15} = 0.043445.
\]

On the basis of the formulas (1.5), (1.6) and (2.3) we have

\[
\tau(r) = \frac{4G_0}{\pi} \sum_{n=0}^{3} \left( \frac{\sqrt{1-x^2}}{m1} \sum_{k=m}^{3} \frac{\alpha_k + i\beta_k}{(k-m)!} + O(\lambda^{-8}) \right) \left( x = \frac{r}{a} \right);
\]

\[
M_0 = 16G_0 a^3 \sum_{k=0}^{3} \frac{\alpha_k + i\beta_k}{2k+3} + O(\lambda^{-8}).
\]

Here

\[
\alpha_0 = 1 + a_2 \lambda^{-2} + a_4 \lambda^{-4} + a_{12} \lambda^{-6}; \quad \alpha_1 = a_1 \lambda^{-1} + a_5 \lambda^{-4} + a_{11} \lambda^{-6};
\]

\[
\alpha_2 = a_3 \lambda^{-4} + a_9 \lambda^{-6}; \quad \alpha_3 = a_3 \lambda^{-3}; \quad \beta_0 = a_6 \lambda^{-3} + a_9 \lambda^{-5} + a_{13} \lambda^{-7};
\]

\[
\beta_1 = a_7 \lambda^{-5} + a_{11} \lambda^{-7}; \quad \beta_2 = a_{13} \lambda^{-7}; \quad \beta_3 = 0.
\]