

\section{Introduction}

Owing to the considerable energy dissipation in viscoelastic materials under variable loading, low thermal conductivity, and sizable temperature dependence of the physicomechanical properties in products made from polymers subjected to periodic actions, we can observe a noticeable increase in temperature, which proves to have a significant effect on the efficiency. The problem of vibrational heat production is important in the investigation of the efficiency of solid-fuel engines \cite{7}, the durability of rubber-and-metal elements, in particular, shock absorbers \cite{6, 9}, the efficiency of various types of vibration-protection systems of rod type, plate type, and shell type \cite{5, 20}, the carrying capacity of fiberglass products \cite{8}, etc. Of special importance is the problem of the determination of the critical parameters; an increase in the value of these parameters leads to a sharp increase in the temperature, which causes the destruction of the structure because of the softening of the material. For a broad class of materials the attainment of a critical thermal state can serve as a criterion for the failure of the carrying capacity of the part \cite{13}.

In many cases an investigation of the critical thermal states reduces to a solution of the nonlinear boundary-value problem (1.1)

\begin{equation}
\begin{aligned}
\nabla^2 u + \lambda g(x) \Phi(u) &= 0 & \text{in } V; \\
\frac{\partial u}{\partial n} + Bh(x)u &= 0 & \text{on } S.
\end{aligned}
\end{equation}

Here \( V \) is a closed region bounded by the surface \( S \); \( u \) is a scalar function; \( \Phi(u) \) is a nonlinear source; \( \lambda \) is a parameter that determines the level of intensity of the sources; \( h(x) \) is a positive, piecewise-smooth function; and \( g(x) \) is a positive, twice continuously differentiable function.

This problem is also encountered in the theory of steady-state thermal detonation \cite{10}; in the study of temperature fields of viscous liquids, gases, and gaseous mixtures \cite{15}; in conductors and electrolytes when an electric current is passed through them (Joule heat) \cite{16}; in the investigation of internal heating of glaciers \cite{1}; etc. Of greatest interest are the limiting values \( \lambda^* \) of the parameter \( \lambda \), above which positive solutions of the problem (1.1) do not exist.

In the studies \cite{2-5, 7, 8, 13, 20} certain problems of the determination of critical parameters in viscoelastic media are considered for vibrational loading for specific functions \( \Phi(u) \).

Recently, a number of investigations have appeared on the development of general methods of calculation of temperature fields described by a boundary-value problem of the form (1.1). A review of mathematical investigations on the use of the method of quasilinearization in systems of the form

\begin{equation}
L^1 u = \lambda^1 \Phi(u) \quad \text{in } V; \\
Lu = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} \left[ a_{ij}(x) \frac{\partial u}{\partial x_j} \right] + a_0(x)u; \\
a_0(x) > 0, \quad \sum_{i,j=1}^{n} a_{ij}(x) \xi_i \xi_j \geq a \geq 0, \quad \| \xi \| = 1
\end{equation}

for the boundary conditions

\[ Bu = \alpha(x) + \beta(x) \sum_{i,j=1}^{n} n_i(x) a_{ij}(x) \frac{\partial u}{\partial x_i} = 0 \text{ on } S \]

is represented in the studies [12, 19].

In [19, 18] a perturbation method is proposed for the investigation of nonlinear problems in eigenvalues of form (1.1). However, in them the existence of a trivial solution \( u_0 \) is proposed, which is equivalent to the equality \( \Phi(u_0) = 0 \). Sources of heat appearing in viscoelastic media do not satisfy this requirement.

In [15, 16] for the bounded function \( u/\Phi(u) \) we are given an upper estimate \( \lambda_* \) for \( \lambda \). For \( \lambda > \lambda_* \), a positive solution of the boundary-value problem (1.1) does not exist. It is shown [15, 16] that

\[ \lambda_* = \mu_0 \max_{u \geq 0} \frac{\mu}{\Phi(u)} \]

(1.3)

where \( \mu_0 \) is the minimum eigenvalue of the auxiliary linear problem for the eigenvalues

\[ \nu^2u + \mu g(x) u = 0 \quad \text{in } V; \]

\[ \frac{\partial u}{\partial n} + Bh(x) u = 0 \quad \text{on } S. \]

(1.4)

The function (1.3) gives a very good estimate of the critical parameter \( \lambda \), especially for small \( \beta \); however, this very method for obtaining Eq. (1.3) does not allow the obtained result to be refined. Moreover, this equation leads to a considerable error in the determination of the values of the unknown function for \( \lambda = \lambda_* \).

In the present study we propose two methods for solving the problem (1.1) with a significantly nonlinear source \( \Phi(u) \), which enables us to obtain with any degree of accuracy the value of \( \lambda \). Essentially, both methods – the small-parameter method and the successive-approximations method – are generalizations of the results of the studies [12, 15, 16, 18], which, in turn, generalize the results of Poincaré to the boundary-value problems for partial differential equations.

We consider specific examples of an investigation of the thermal instability of viscoelastic elements subjected to cyclic loading. We indicate the possibility of using the proposed methods for solving other physical problems.

In particular, we study the static stability of a bar with initial imperfections on a nonlinear elastic base.

§ 2. The Small-Parameter Method (SPM)

Assuming that the function \( u/\Phi(u) \) is bounded and that it has a form such that the estimate (1.3) is sufficiently exact, we propose the following approach to the solution of the system (1.1). We introduce a fictitious, small parameter \( \varepsilon \), which eventually is assumed equal to unity, and we write problem (1.1) in the form

\[ \nabla^2u + \mu g(x) u = \varepsilon g(x) [\mu u - \lambda \Phi(u)] \quad \text{in } V; \]

\[ \frac{\partial u}{\partial n} + Bh(x) u = 0 \quad \text{on } S, \]

(2.1)

(2.2)

where \( \mu \) is an arbitrary parameter.

For the selection of \( \mu \), we take into account that the mathematical assertion of [17] holds: let \( \rho(x) \) be a function that is positive and continuous in \( V \) and let \( \Phi(x) \) be a twice continuously differentiable function that satisfies the inequality

\[ L \varepsilon - \lambda \rho(x) \varepsilon \geq 0 \quad \text{in } V; \]

\[ B \varepsilon = 0 \quad \text{on } S. \]

(2.3)

Here \( L \) is defined by Eq. (1.2).

Then, \( \Phi(x) \geq 0 \) in \( V \) if and only if \( \lambda < \mu_0 \), where \( \mu_0 \) is the minimum eigenvalue of the problem

\[ L\psi - \mu_0 \psi = 0 \quad \text{in } V; \]

\[ B\psi = 0 \quad \text{on } S. \]

(2.4)