DETERMINATION OF THE STRESSES IN AN ELASTIC CYLINDER COMPRESSED BY TWO PUNCHES

M. I. Teplyi

§1. We consider the problem of the stressed state of an elastic cylinder (disk), which is in contact with two identical rigid cups (punches) with curvilinear profile of the contact surfaces, the radii of curvature of which are several times greater than the radius of the cylinder. Let a disk $S_2$ of unit thickness and radius $R_2$ be compressed by forces $P$ through the punches $S_1$ (Fig. 1), and let us assume that a torque $M_0$ acts at the center of the disk. If the radius of curvature $R_1$ of the surface of the cup that is in contact with the disk differs little from the radius of the disk $R_2$, then the contact region $L_1 + L_2$ between the bodies $S_1$ and $S_2$ during their compression will be commensurate with the radii $R_1$ and $R_2$. As a result of the action on the disk of the indicated force factors on the contact surface $L_1 + L_2$, there arise normal stresses $\sigma_T$ and tangential stresses $\tau_{\varphi \theta}$, which must be determined. The problem also consists of finding the components of the stress tensor at an arbitrary point of the region occupied by the disk.

We introduce a system of rectangular Cartesian coordinates $xOy$ such that the origin of this system coincides with the center of the disk, and the $Ox$ axis coincides with the line of action of the principle vector of the external forces $P$. We denote by $\alpha$ and $\vartheta$ the polar angles of the points of the boundary of the disk $S_2$ (see Fig. 1).

In agreement with the formulation of the problem, on the contour of a disk we have the boundary conditions

$$\sigma_{\varphi}(\alpha) = \tau_{\varphi \theta}(\alpha) = 0 \quad \text{for} \quad -\alpha_0 \leq \alpha \leq \alpha_0; \quad -\alpha_0 \geq \alpha \geq \pi + \alpha_0; \quad \alpha_0 \geq \alpha \leq \pi + \alpha_0; \quad \pi - \alpha_0 \leq \alpha \leq \pi + \alpha_0,$$  

where $\alpha_0$, $-\alpha_0$, $\pi - \alpha_0$, $\pi + \alpha_0$ are the polar angles of finite points of, respectively, the contact regions $L_1$ and $L_2$; $k_1(\alpha)$ and $k_2(\alpha)$ are the curvatures of the boundaries of the bodies $S_1$ and $S_2$ in the contact regions.

Having used the method of Mushkelishvili [2], and also the results of [3, 4], on the basis of the boundary conditions (1.1) and (1.2) for the case of a circular notch of the punch [$k_1(\alpha) = 1/R_1$] we obtain an integrodifferential equation for the problem

$$\frac{1}{\pi} \int_{-\alpha_0}^{\alpha_0} \cot g(\alpha - \theta) [\sigma'(\theta) + \tau_{\varphi \theta}(\theta)] d\theta - \frac{\alpha_0 - 1}{\alpha_0 + 1} [\sigma(\alpha) - \tau_{\varphi \theta}(\alpha)] = -\frac{2E_2}{(\alpha_0 + 1)(1 + \nu_2)R_1} (-\alpha_0 \leq \alpha \leq \alpha_0).$$  

Here $\sigma' = d\sigma/d\alpha; \tau_{\varphi \theta} = d\tau_{\varphi \theta}/d\alpha; \varepsilon = R_1 - R_2; E$ is the elastic modulus of the material of the disk; $\nu_2 = 3 - 4\nu$ for a plane deformation; $\alpha_0 = 3 - \nu_2/1 + \nu_2$ for a generalized plane stressed state; and $\nu_2$ is the Poisson ratio.
We consider certain particular cases of Eq. (1.3). In the contact region, let there arise frictional forces given by Coulomb's law, i.e., \( \tau_{r,q} = \mu \sigma \) (\( \mu \) is the coefficient of sliding friction).

In this case, Eq. (1.3) takes the form

\[
\frac{1}{\pi} \int_{-\alpha_a}^{\alpha_a} \cos (\alpha - \phi) \frac{p'(\phi) + \lambda p(\phi)}{\cos \alpha} d\phi = \frac{2\pi E_2}{(x_2 + 1)(1 + \nu_2) R_1} \left( -\alpha_a \ll \alpha \ll \alpha_a \right),
\]

(1.4)

where \( p(\alpha) = -\sigma_\theta(\alpha) \) is the normal pressure in the contact region; \( p'(\alpha) = dp(\alpha)/d\alpha \).

Furthermore, the equilibrium conditions for a disk imply

\[
P = \int_{-\alpha_a}^{\alpha_a} \cos \alpha \rho(\alpha) d\alpha.
\]

(1.5)

When the frictional forces in the contact region can be neglected, i.e., when \( \lambda = 0 \), from (1.4) and (1.5) we obtain

\[
\frac{1}{\pi} \int_{-\alpha_a}^{\alpha_a} \cos (\alpha - \phi) p'(\phi) d\phi = \frac{2\pi E_2}{(x_2 + 1)(1 + \nu_2) R_1};
\]

(1.6)

\[
P = \int_{-\alpha_a}^{\alpha_a} p(\alpha) \cos \alpha d\alpha.
\]

(1.7)

An equation similar to (1.6) was obtained in [1].

§2. Equations (1.3), (1.4), and (1.6) are integrodifferential equations, the solution of which encounters considerable mathematical difficulty. Therefore, the question of an approximate solution arises.

We consider Eq. (1.4), corresponding to the case in which in the contact region between the punches \( S_1 \) and the disk \( S_2 \) there arise forces of sliding friction, and we find an approximate solution. In order to do this, we represent the unknown approximate solution of Eq. (1.4) by the expression

\[
p(\alpha) = \frac{\pi E_2}{R_1} \sqrt{\alpha_0^2 - \xi^2} \sum_{n=0}^{2N} c_n \xi^n \left( -\alpha_a \ll \xi \ll \alpha_a \right),
\]

(2.1)

where the \( c_n \) are unknown coefficients, \( \xi = \tan \alpha, \alpha_0 = \tan \alpha_0 \).

We substitute Eq. (2.1) into (1.4), and we require that the coefficients \( c_n \) satisfy this equation at a finite number of equidistant points \( \xi_j \) (\( j = 0, \pm 1, \pm 2, \ldots, \pm N \)). Assuming \( \xi_j = j\pi/(N+1) \), we obtain the following system of equations for finding the coefficients \( c_n \):

\[
\sum_{n=0}^{2N} c_n A_{n,j}(\xi_i) = \frac{2\pi E_2}{(x_2 + 1)(1 + \nu_2) R_1},
\]

(2.2)