CRITICAL LOADS FOR NON-MILDLY-SLOPING ELLIPSOIDAL SHELLS OF UNEQUAL AXES UNDER PRESSURE

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1. Closed Ellipsoidal Shells under a Uniformly Distributed Pressure \( p \)

It has been shown in \([1, 2]\) that loss of stability close to a zero-moment stressed equilibrium state of a thin linearly elastic isotropic highly convex shell with the formation of dents which are not adjacent to the edge is possible if the equality

\[
\max_{(\sigma)} \frac{-Z + [Z^2 + 4K (T^{12}T^{22} - T^{11}T^{22})]^\frac{1}{2}}{K} = e
\]

is attained. Here \( e = 2E\delta^2/\sqrt{3(1-\nu^2)} \); the maximum is taken with respect to all the interior points of the medial surface \( F \) of the shell under discussion; \( K \) is the Gaussian curvature of \( F \); \( Z \) is the surface load acting on the shell; \( T^{\alpha\beta} \) are the stresses caused in the shell by the action of an external load in a subcritical equilibrium state, which are determined from the linear zero-moment theory; \( E \) is Young's modulus; \( U \) is the Poisson coefficient; and \( \delta \) is the thickness of the shell.

At the same time let us assume for the case in question of a uniformly distributed pressure \( Z = -p = const \) that \( p > 0 \) corresponds to an external pressure. Equation (1.1) determines the main asymptote of the upper critical load in the case of an exceedingly small shell thickness \([2]\).

Let us refer the medial surface \( F \) of the ellipsoidal shell to the curvilinear coordinates \( \xi_1 = \theta, \xi_2 = \varphi \) so that the Cartesian coordinates \( x^i \) of the points are specified in the following parametric form:

\[
x^1 = a_1 \sin \theta \cos \varphi; \quad x^2 = a_2 \sin \theta \sin \varphi; \quad x^3 = a_3 \cos \theta
\]

\((0 \leq \varphi \leq 2\pi, 0 \leq \theta \leq \theta_0)\). (1.2)

The equation \( \varphi = \theta_0 \) defines the edge \( \partial F \) of the ellipsoid \( F \). When \( a_1 = 1 \), the ellipsoid \( F \) changes into a sphere \( F \), the Cartesian coordinates of whose points will be \( x^i = x^i/a_1 \). Without any loss in generality, let us select the numbering of the semiaxes \( a_i \) such that

\[
a_1 > a_2.
\]

(1.3)

In addition let us assume in Sec. 1 that \( a_2 = a_3 \). The Gaussian curvature of the ellipsoid is found from the formula

\[
K = \left[ a_1a_2a_3 \sum_{k=1}^{3} b_k(x^k)^2 \right]^{-2},
\]

(1.4)

where \( b_k = 1/a_k^2 \).

The stresses \( T^{\alpha\beta} \) caused in the ellipsoidal shell by the action of the pressure \( p \) are determined with the use of the method of affine transformations in the linear zero-moment theory of shells \([3]\), pp. 175-184). The final expressions for the stresses assume the following form:

\[
T^{11} = \tilde{T}^{11}; \quad T^{12} = \tilde{T}^{12}; \quad T^{22} = \frac{1}{\tilde{T}} \tilde{T}^{22};
\]

(1.5)
Here
\[ \alpha, \beta = 1, 2; \ T^2_1 = -b_3 - (b_2 - b_1) \cos 2\varphi; \ T^2_0 = T^2_{11} = - (b_2 - b_1) \cos \theta \sin 2\varphi; \]
\[ T^2_2 = -b_3 - (b_2 + b_1 - 2b_3) \sin^2 \theta - (b_2 - b_1) \cos^2 \theta \cos 2\varphi; \]
\[ t^2_{11} = -t^2_{22} = \cos 2k \varphi; \ t^2_{12} = t^2_{21} = -\sin 2k \varphi; \]
\[ t^2 = \frac{(a_1^2 \cos^2 \varphi + a_2^2 \sin^2 \varphi) \cos^2 \theta + a_3^2 \sin^2 \varphi}{a_1^2 \sin^2 \varphi + a_2^2 \cos^2 \varphi}; \]

(1.6)

For a closed ellipsoidal shell \( \theta_0 = \pi \). It follows from the requirement that the stresses \( T^{\alpha \beta} \) be bounded at \( \theta = \pi \) that
\[ \lambda_k = 0 \quad (k = 1, 2, \ldots). \]

(1.8)

for a closed shell.

Let us substitute (1.5)-(1.8) into Eq. (1.1). Having found the maximum in (1.1), we arrive at the following results.

The main asymptote of the upper critical load \( p_0 \) for a sufficiently thin closed ellipsoidal shell under external pressure is determined by the formula

\[ p_0 = e - b_1 b_2 \]

(1.9)

The loss of stability is accompanied in this case by the formation of dents in the neighborhood of the points of \( F \) which lie on the smallest semiaxis \( (x_1 = x_2 = 0) \), since the maximum in (1.1) is attained at these points [1, 2].

Loss of stability is possible in the case of internal pressure \((p < 0)\) if

\[ b_3 - b_2 - b_1 > 0. \]

(1.10)

In this connection the main asymptote of the upper critical load is

\[ |p_0| = \frac{e b_1 b_2}{b_3 - b_2 - b_1}, \quad \text{if} \quad 0 < b_3 - b_2 - b_1 \leq 2b_2; \quad |p_0| = \frac{8eb_1 b_2}{(b_3 + b_2 - b_1)^2}, \quad \text{if} \quad 2b_2 \leq b_3 - b_2 - b_1. \]

(1.11)

The loss of stability is accompanied by the appearance of dents in the first case in the neighborhood of points of the ellipsoid \( F \) which lie on the intermediate semiaxis \( (x^1 = x^2 = 0) \), and in the second case in the neighborhood of points with the coordinates

\[ x^1 = 0; \quad |x^3| = a_3 \frac{b_3 - 3b_2 - b_1}{5 (b_3 - b_2)}. \]

(1.12)

2. Free Resting against the Plane of Symmetry

Let us consider an ellipsoidal shell (1.5) with edge \( \theta = \theta_0 = \pi/2 \) under uniform external pressure. Let the edge of the shell \( \partial F \) be free from the action of the bending moment and stresses which act in the plane of the edge; in addition, displacements which are orthogonal to the plane of the edge are equal to zero. Then (21, Sec. 7) there is possible, along with the possibility of stability loss accompanied by the formation of dents not adjacent to \( \partial F \), a loss of stability in the case of the loads (1.1) which is associated with large deformations (bulging) of the edge \( \partial F \) and its neighborhood on \( F \) in the case of loads determined by the equation [(21, Eq. (7.11))]

\[ \max_{(\partial F)} \left( \frac{2T_{11}}{eb_{22}} \right) = \frac{1}{2}, \]

(2.1)

where \( b_{22} \) is the normal curvature of \( F \) in the direction of \( \partial F \),

\[ b_{22} = -a_1 a_2 (a_1^2 \sin^2 \varphi + a_2^2 \cos^2 \varphi)^{-\frac{3}{2}}. \]

(2.2)