NUMERICAL SOLUTION OF STATIC BOUNDARY-VALUE PROBLEMS FOR AXISYMMETRICAL SHELLS BY REDUCTION TO CAUCHY PROBLEMS

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Computational errors in the method of reduction of boundary-value problems to Cauchy problems lead to unsatisfactory results in the case of long shells [1]. For this reason the pivotal method has been used [1], and the method of discrete orthogonalization [4] has been used in [2, 3]. In the present study we describe the computational difficulties incurred in solving the problem by conventional means and suggest some modifications of the method of solution that are based on physical considerations and yield better results.

The stated problem is described by a set of ordinary differential equations in canonical form. The unknowns entering into it are the bending moment $M_1$ acting in the cross section, the radial (H) and axial (V) forces, and their corresponding angle of rotation $\theta_1$, radial displacement $u$, and displacement $w$ along the rotation axis (Fig. 1). Considering the diverse character of the unknowns, we transform to the appropriate dimensionless and normalized variables given by the formulas

$$
\tilde{M}_1 = \frac{M_1 \sqrt{12}}{q_0 e_0^2 p_0}; \quad \tilde{H}, \tilde{V} = \frac{H, V}{q_0 e_0}; \quad \tilde{\theta}_1 = \frac{\partial D_0 \sqrt{12}}{q_0 e_0^2 p_0}; \quad \tilde{u}, \tilde{w} = \frac{u, w}{q_0 r_0^2}.
$$

Here $r_0$, $h_0$, $q_0$, and $E_0$ are the values chosen in each specific problem for the linear dimension of the median surface of the shell, its thickness, the load, and the elastic modulus, and

$$
D_0 = \frac{E_0 h_0}{12(1-\mu^2)}; \quad B_0 = \frac{E_0 h_0}{2(1-\mu^2)}.
$$

The normalizing factors are chosen to make the actual values of the variables have an order of magnitude close to unity. Hereinafter we treat equations in $\tilde{M}_1 \tilde{r}$, $\tilde{H} \tilde{r}$, $\tilde{V} \tilde{r}$, $\tilde{\theta}_1$, $\tilde{u}$, $\tilde{w}$ ($\tilde{r} = r/r_0$). Regarding these variables as components of a vector $z$, we write the set of equations in the form

$$
\frac{dz}{ds} = F(s) z + f(s),
$$

where $F(s)$ and $f(s)$ are a known matrix and a known vector.

This set of equations has been given in scalar form in [5]. For simplicity we denote the components of the vector $z$ by $M_1$, H, V, $\theta_1$, u, and w.

The boundary conditions for the kind of problems normally met in practice entail specification of three of the given unknowns at the shell boundaries $a$ and $b$.

The set of equations (2) is distinguished by the fact that the unknown axial force V is determined only by its initial value and the external load, while the axial displacement $w$ does not occur in the right-hand sides of the system. These special features enable us to reduce the given class of problems to one in which four unknown components of $z$, including the axial force and axial displacement, are specified at the left boundary, while only two components other than the axial force and axial displacement are given at the right boundary.
In the usual boundary conditions the axial displacement \( w \) is given only at one boundary of the shell, and the axial displacement, therefore, is given at the other boundary. For example,
\[
\omega_a = \omega_a^*; \quad V_b = V_b^*.
\]
In this case the problem is equivalent to the fundamental problem. Thus, the force \( V_a \) at the boundary \( a \) is readily determined from the equilibrium equation for the entire shell. Conversely, at the boundary \( b \) we can set \( w_b^0 = 0 \). The solution corresponding to this additional condition is correct with the exception of \( w_n \), which differs from the true value \( w \) by a constant: \( w = w_n + c \). The latter is obviously evaluated from the condition
\[
\omega_a^0 + c = \omega_a^*.
\]
In the event that the axial displacement is given at both ends of the shell we multiply the given external effects, including the nonzero boundary values, by an unknown parameter \( \psi \). We discard the boundary condition \( w_b = w_b^* \) and retain all the others. We analyze this problem for different values of the axial force \( V_a \) and the parameter \( \psi \). The variable \( w_b \) in this case is linear and a homogeneous function of \( V_a \) and \( \psi \), namely:
\[
w_b = w_b(V_a, \psi),
\]
whence we find
\[
w_b = w_b(0, \psi) + V_a w_b(1, 0).
\]
Setting \( w_b = w_b^* \) and \( \psi = 1 \) in the real problem, we obtain
\[
V_a = \frac{V_a w_b^*(0, 1)}{w_b^*(1, 0)}
\]
We see that the reduction to the fundamental problem when the displacement is given at both ends requires the solution of two auxiliary fundamental problems.

We now consider the solution of the fundamental boundary-value problem for the unknown six-component vector \( z \). We denote by \( y_a \) the four-component vector specified on the left, and by \( y_b \) the two-component vector specified on the right. We denote the respective vectors formed by the left and right unknowns by \( x_a \) and \( x_b \). The equations and boundary conditions are written as follows in our notational convention:
\[
\frac{dz}{ds} = F(s) z + \psi f(s); \quad y_a = \psi x_a; \quad y_b = \psi x_b.
\]
We now describe a modified method of reduction of the given problem to Cauchy problems. We temporarily discard the boundary conditions at \( a \) and \( b \) and consider the vectors \( x_a, x_b \) and the parameter \( \psi \) to be variables. Then Eqs. (4) determine \( y_b \) as a linear homogeneous function of those variables, and the value of that function is found by solution of the Cauchy problems.

We represent the function \( y_b \) in the form
\[
y_b = y_b(x_a, y_a, \psi).
\]
Due to linearity and homogeneity we have
\[
y_b = y_b(0, y_a, \psi) + y_b(x_a, 0, 0).
\]