THE THREE-DIMENSIONAL CONTACT PROBLEM FOR A ROUGH DIE

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Plane contact problems, taking account of the forces of friction between a die and an elastic half-plane have been studied in rather great detail [1, 3]. Very little attention is being paid in the literature to the analogous three-dimensional problems. The axisymmetric problem was discussed in [2]. The unsymmetrical problem is obviously considered for the first time.

§1. Let a die, round in a plan view, the equation of whose surface $z = f(\rho, \psi)$, be indented by the axial force $P$ into a transversally isotropic half-space $z \geq 0$. A force equal to $kP$, where $k$ is the friction coefficient, also acts on the die in the direction of the Ox axis.

We assume that the contact region is a circle of radius $a$, and that the tangential stress $\tau_{zx}$ at every point is proportional to the normal pressure $\sigma$. We neglect the value of $\tau_{yz}$. Then, the principal integral equation of the problem under consideration assumes the form [4]

$$\int_0^{2\pi} \int_0^\infty \frac{\sigma(\rho, \psi)}{R} \rho d\rho d\psi = \frac{w}{H} - \alpha k \Re \int_0^{2\pi} \int_0^\infty \frac{\sigma(\rho, \psi)}{\rho^{1/4} - \rho^{1/4}} d\psi.$$  \hspace{1cm} (1.1)

Here $w(\rho, \psi)$ is the normal displacement in the contact zone; $H, \alpha$ are constants, determined by the elastic properties of the material of the half-space.

For convenience, we write Eq. (1.1) in operator form

$$L\sigma = \frac{w}{H} - \alpha k M\sigma,$$  \hspace{1cm} (1.2)

where $L$ and $M$ are, respectively, the integral operators standing in the left- and right-hand parts of Eqs. (1.1).

For real bodies, the values of $\alpha (0 < \alpha < 0.5)$ and $k$ are less than unity; therefore, it is convenient to adopt their product as a small parameter. Let the following expansion hold

$$\sigma(\rho, \psi) = \sum_{n=0} \alpha^k \sigma_n(\rho, \psi).$$  \hspace{1cm} (1.3)

Substituting expression (1.3) into (1.2) and equating terms with identical powers of the small parameter $\alpha k$, we obtain an infinite system of integral equations

$$L\sigma_0 = \frac{w}{H}; \quad L\sigma_1 = -M\sigma_0; \ldots; \quad L\sigma_n = -M\sigma_{n-1}.$$  \hspace{1cm} (1.4)

The operator $L^{-1}$, inverse to $L$, is known [5]. For the given case it has the form

$$L^{-1}f = \frac{1}{2\pi^2} \int_0^{2\pi} \int_0^\infty \frac{f(\rho, \psi) + \Phi(\rho, \psi, \psi)}{V^{1/4} - \rho^{1/4}} \rho d\rho d\psi - \int_\rho^{2\pi} \frac{d\Phi(r, \rho, \psi, \psi)}{V^{1/4} - \rho^{1/4}} d\psi;$$
Formulas (1.5) permit a complete solution of the problem posed in quadratures

\[ \sigma = \sum_{n=0}^{\infty} (ak)^n (-L^{-1}M)^n L^{-1} \left[ \frac{w(r, \psi)}{H} \right] . \]

§2. As an example, let us consider the case of a flat round die. Under the action of normal and tangential forces, let the die be indented by the amount \( \omega_0 \), and let it be rotated around the Oy axis by the angle \( \delta \). Then

\[ \omega(p, \psi) = \omega_0 + \delta \rho \cos \psi . \quad (2.1) \]

The condition for the close fit of the bottom of the die to the half-space has the form

\[ \omega_0 - 6a > 0 \quad (\delta > 0) . \]

Substitution of (2.1) into the first equation of (1.4), after the use of formulas (1.5), gives

\[ \sigma_0 = \frac{1}{\pi H} \frac{\omega_0 + 2\delta \rho \cos \psi}{\sqrt{a^2 - \rho^2}} . \quad (2.2) \]

Substituting (2.2) into the second equation of (1.4) and solving it, we find a term of the series

\[ \sigma_1 = \frac{2}{\pi H} \left\{ \frac{a\ln \left[ \frac{1 - \rho^2}{a^2 - \rho^2} \right]}{\rho \sqrt{a^2 - \rho^2}} \cos \psi \right. \\
+ \delta \left[ \frac{1}{\rho} \frac{d}{dp} \frac{a}{\rho \sqrt{a^2 - p^2}} \ln \sqrt{\frac{a + x}{a - x}} + \frac{a}{\sqrt{a^2 - p^2}} \right] \\
+ \left( \frac{a^2 \ln \left[ \frac{1 - \rho^2}{a^2 - \rho^2} \right]}{\sqrt{a^2 - \rho^2}} + \frac{a}{2 \sqrt{a^2 - \rho^2}} \right) \cos 2\psi \right\} . \quad (2.3) \]

The procedure can be continued and, in the limit, an exact solution of the problem can be obtained.

For the subsequent investigation, we shall limit ourselves to finding a two-term approximation. The principal vector and the principal moment of the above system of stresses are determined from the formulas

\[ P = \frac{2a}{\pi H} \left( \omega_0 + \frac{ak}{\pi} \delta b \right); \quad M_0 = \frac{2a^2}{\pi H} \left( \frac{2}{3} a \delta - \frac{ak}{\pi} \omega_0 b \right) . \quad (2.4) \]

If the point of application of the axial force is turned with respect to the Oy axis by the amount \( b \), then, \( M_0 = Pb \), and formulas (2.4) establish a connection between the angular and axial displacements of the die

\[ \delta = \frac{b + \frac{ak}{\pi} a}{\frac{2}{3} a - \frac{ak}{\pi} b} \omega_b . \quad (2.5) \]

Relationship (2.5) shows that the die is inclined even when the external forces do not create a tilting moment. In order for there to be no angular displacement of the die, the point of application of the axial force must be shifted toward the side opposite to the direction of the tangential force, by an amount \( ak \pi / \pi \). An analogous effect is observed also in plane contact problems [3].

Formulas (2.4) can be regarded as rather exact, since taking account of the third term of the series (1.6) leads to the following expression for the value of the principal vector:

\[ P = \frac{2a}{\pi H} \left[ \omega_0 \left( 1 - \frac{ak^2}{6} \right) + \frac{ak}{\pi} \delta \right] . \quad (2.6) \]

In view of the smallness of the factor \( \alpha^2 k^2 \), the correction introduced by the third term can be considered insignificant.