FLEXURE OF RECTANGULAR THICK PLATES WITH ARBITRARY BOUNDARY CONDITIONS

A. M. Voin

Prikladnaya Mekhanika, Vol. 3, No. 8, pp. 11-18, 1967

UDC 593.3

In the present paper the symbolic method of writing solutions of equations of the theory of elasticity, proposed by A. I. Lur'e [3, 4], is developed for the case of flexure of a thick rectangular plate with arbitrary boundary conditions. In the general case the problem is divided into four parts, and the solution of each of these is reduced to an infinite system of algebraic equations. The coefficients of the system depend on the relationships between the characteristic dimension of the plate and its material, but the form of the load and the boundary conditions affect only the right sides of the equations.

1. As is known, A. I. Lur'e [3, 4] reduced the three-dimensional flexure problem of a thick plate to a two-dimensional problem by expressing the displacements $u$, $v$, and $w$ in terms of functions $u_0$, $v_0$, and $w_0$ of two variables (the antisymmetric flexure problem is considered, which in [3] is called problem B; however, by appropriate modifications the method can be extended to the symmetric problem).

The functions $u_0$, $v_0$, and $w_0$ satisfy the equations

$$
\begin{align*}
\frac{d}{dz}u_0 + \frac{d}{dz}v_0 + \frac{d}{dz}w_0 &= \frac{q_1}{G}, \\
\frac{d}{dz}u_0 + \frac{d}{dz}v_0 + \frac{d}{dz}w_0 &= \frac{q_2}{G}, \\
\frac{d}{dz}u_0 + \frac{d}{dz}v_0 + \frac{d}{dz}w_0 &= \frac{q_3}{G},
\end{align*}
$$

where $q_1$, $q_2$, and $q_3$ are the components of the external load at the ends of the plate $z = h$ and $z = -h$ in problem B, directed, respectively, along the $x$-, $y$-, and $z$-axes; $d_{ik}$ are the differential operators given by the right sides of formulas (2.22) in reference [3], with $z$ in them replaced by $h$.

The methods of determining particular solutions of system (1.1) have been worked out in the monograph [3] for a broad class of loads.

The homogeneous solutions, as shown by A. I. Lur'e, have the form

$$
\begin{align*}
\phi &= Q_{ik} \varphi, \\
\psi &= Q_{ik} \psi, \\
\psi &= Q_{ik} \psi,
\end{align*}
$$

Here $Q_{ik}$ is the co-factor of the element $d_{ik}$ of the determinant consisting of the operator coefficients of system (1.1); the function $\varphi$ satisfies one of the three equations

$$
D^{(2)} \varphi = 0,
$$

where $k_i$ are the roots of the transcendental equation $\sin 2k - 2k = 0$, and $k_j$ are the roots of the equation $\cos k = 0$.

The functions $\varphi_1$, $\varphi_j$, and the numbers $k_i$, $k_j$ are respectively called eigenfunctions and eigenvalues.

The roots of the equation $\sin 2k - 2k = 0$ are complex and are grouped in fours with equal moduli; the zero root corresponds to the biharmonic solution. In the following we use roots with a positive real part.

The roots of the equation $\cos k = 0$ are real and have the form $k_j = \pm \pi (2j + 1)/2$, ($j = 0, 1, 2, \ldots$). Thus, we have three infinite sets of homogeneous solutions (the complex solutions correspond to the two real solutions); by these we satisfy on the side surface of the plate the same number of conditions. The latter are obtained by expanding the initial conditions on the boundary of the plate in series of Legendre polynomials along the $z$-axis, and equating the coefficient of the same terms.

The formulation of the boundary conditions by means of Legendre polynomials corresponds to the physical nature of the problem, since by considering only the zeroth and first moments of the loads on the side surface we fulfill the equilibrium conditions. If displacements are specified on the side surface, then, confining ourselves only to the zeroth and first moments of them, we can talk of considering "equilibrium" in an analogous sense. Here it is meant that the classical principle of Saint-Venant can be extended to displacements, which often is done in implicit manner.

2. The displacements of points of the middle surface corresponding to the $i$-th eigenvalue are obtained from formulas (1.2) with the use of the operators $Q_{ik}$, $Q_{ik}$, and $Q_{ik}$. The choice of the latter is governed by the form of the final results, which is more convenient for calculations in comparison with the operators $Q_{ik}$, $Q_{ik}$, and $Q_{ik}$ [3].

We introduce the notations

$$
\begin{align*}
\delta_i &= 1 - \frac{m \sin^2 k_j}{2(m - 1)^2}, \\
\lambda_i &= \frac{1}{2(m - 1)}(m \cos^2 k_j - 3m + 2), \\
\eta_i &= \frac{1}{2(m - 1)}(m \cos^2 k_j + m - 2), \\
\kappa &= \frac{m \cos^2 k_j}{2(m - 1)} - 1,
\end{align*}
$$

Then for the functions $u_{ij}$, $v_{ij}$, $w_{ij}$, corresponding to the eigenfunctions $\varphi_i$, we obtain the formulas

$$
\begin{align*}
u_i &= \frac{h \lambda_i \varphi_i}{h \eta_i \varphi_i}, \\
w_i &= \frac{h \lambda_i \varphi_i}{h \eta_i \varphi_i},
\end{align*}
$$

and for the functions $u_{ij}$, $v_{ij}$, and $w_{ij}$, corresponding to $\varphi_j$, we have the expressions

$$
\begin{align*}
u_i &= -2h \lambda_i \varphi_j, \\
v_j &= -2h \lambda_i \varphi_j, \\
w_i &= 0.
\end{align*}
$$

In deriving them we used Eqs. (1.3). The displacements corresponding to $\varphi_j$ and $\varphi_j$ are found from formulas (2.17) and (2.18) of [3]. They equal

$$
\begin{align*}
u_i &= \frac{h \lambda_i}{k_j} \left( \varphi_j \sin \frac{k_j}{h} - \varphi_j \cos \frac{k_j}{h} \right), \\
v_i &= \frac{h \lambda_i}{k_j} \left( \varphi_j \sin \frac{k_j}{h} - \varphi_j \cos \frac{k_j}{h} \right), \\
w_i &= \frac{h \lambda_i}{k_j} \left( \varphi_j \sin \frac{k_j}{h} - \varphi_j \cos \frac{k_j}{h} \right).
\end{align*}
$$
The stresses for various eigenvalues are obtained from relations (2.4) and (2.5) by means of the generalized Hooke's law.

\[ u_i = \sum_{l=0}^{\infty} U_i^{(2l+1)} (z), \quad v_i = \sum_{l=0}^{\infty} V_i^{(2l+1)} (z), \quad w_i = \sum_{l=0}^{\infty} W_i^{(2l+1)} (z), \]  

(2.6)

where \( p_n \) is a Legendre polynomial of degree \( n \).

Analogously we represent the functions \( u_j, v_j \), and \( w_j \). The quantities \( U_j^{(2l+1)}, V_j^{(2l+1)}, \) and \( W_j^{(2l+1)} \) are defined as follows:

\[ U_j^{(2l+1)} = U_j^{(2l+1)} h^2 \partial \phi, \quad V_j^{(2l+1)} = V_j^{(2l+1)} h^2 \partial \phi, \quad W_j^{(2l+1)} = W_j^{(2l+1)} h, \]

(2.7)

Here \( U_1^{(2l+1)}, V_1^{(2l+1)}, \) and \( W_1^{(2l+1)} \) are the numbers calculated from the formulas

\[ U_1^{(2l+1)} = U_1^{(2l+1)} = 4t + 3, \quad V_1^{(2l+1)} = V_1^{(2l+1)} = 4t + 3, \quad W_1^{(2l+1)} = W_1^{(2l+1)} = 4t + 3, \]

(2.8)

The remaining polymoments of the biharmonic function are 0.

3. In the following we assume that the \( x \) and \( y \) coordinates vary within the limits \( a/2 \leq x \leq a/2, \) \( -b/2 \leq y \leq b/2, \) \( a = 1, b = 1, 2h \leq 1. \)

The eigenfunction corresponding to the \( i \)-th (analogous to the \( j \)-th) eigenvalue is represented in the form

\[ \phi_i = \sum_{n=0}^{\infty} (-1)^n A_n \sin \frac{a}{h} \sin \frac{2nx}{b} + \]

\[ + B_n \sin \frac{a}{h} \sin \frac{2nx}{a} + \]

\[ + C_n \sin \frac{a}{h} \sin \frac{2nx}{b} + \]

(2.9)