A variant of generalization of the kinematic theorem of Koiter is proposed for the case of cyclic action of a nonstationary temperature field. The conditions of a progressive failure (one-sided increase in the plastic strain with each cycle) are considered on the basis of this, in relation to problems characterized by the action of a single parameter: external load and nonstationary temperature field. An analogy with problems of the limiting equilibrium is used. As examples, the conditions of a progressive failure are determined for a thick-walled tube (internal pressure) and for a circular plate (uniform load) clamped along the edge in the presence of thermal changes.

1. In reference [8] a generalization of the kinematic theorem of the theory of adaptability [3] is considered for the case where nonstationary temperature fields act. It is shown that, when applied to these conditions, the equation of Koiter must contain an additional (temperature) term. With the latter taken into account, this equation assumes the form

\[ \int_0^T \left( \int X \delta_\theta \delta t \, dp + \int \rho \delta_\theta \delta t \, dS + \int \dot{\dot{\epsilon}}_{ij} \delta_\theta \delta t \, dv \right) \, dt = 0 \]

(1.1)

Here \( \alpha T \) is the actual thermal strain (a function of the coordinates of a point of the body and the time \( t \)); \( \tau_0 \) is a certain time interval for which the increments of the plastic strain

\[ \Delta \epsilon_{ij} = \int_0^{\tau_0} \dot{\epsilon}_{ij} \, dt \]

(1.2)

form a kinematically possible strain distribution; \( \delta_{ij} \) is the Kronecker delta (\( \delta_{ij} = 1 \) for \( i = j \); \( \delta_{ij} = 0 \) for \( i \neq j \)). The rest of the notations here and in what is to follow are the same as in [3].

In [8] a rational method is proposed, enabling us to specify the distributions of the residual velocities \( \dot{u}_b(\tau) \), the velocities of residual stresses \( \dot{\sigma}_{ij}(\tau) \), and the velocities of plastic strains \( \dot{\epsilon}_{ij}(\tau) \), that satisfy the necessary conditions and are based on the use of the relation

\[ \dot{\epsilon}_{ij} = \frac{1}{2} \dot{\sigma}_{ij} - A_{ij} \dot{\sigma}_{\theta 0} \]

(1.3)

As an example illustrating this method, a determination of the conditions for the occurrence of a sign-alternating flow is presented, for thermal change in a plate clamped along the boundary (external load is absent).

In the given work we consider another variant of the expression of the temperature term of Eq. (1.1). It enables us to find comparatively simply solutions of a certain class of problems that are interesting from the viewpoint of technical applications. Here to a large extent we utilize an analogy with problems of the limiting equilibrium.

2. Let us consider the volume integral entering the left side of Eq. (1.1)

\[ \int_0^T \dot{\epsilon}_{ij} \delta_\theta \delta t \, dv. \]

(2.1)

We use \( \sigma^{(1)}_{ij} \) to denote the stresses found in solving the problem of thermoelasticity for a given distribution of the thermal strains \( \alpha T \). The relation between the stresses and strains here has the form

\[ \sigma_{ij}^{(1)} = A_{ij} \sigma_{\theta 0}^{(1)} + \delta_{ij} \alpha T. \]

(2.2)

On the other hand, the rates of change of the residual stresses can be expressed by means of the relation (1.3)

\[ \dot{\sigma}_{ij}^{(1)} = \dot{\epsilon}_{ij} - \dot{\epsilon}_{ij}. \]

(2.3)

Here \( \dot{\epsilon}_{ij} \) and \( \dot{\epsilon}_{ij} \) are, respectively, the kinematically possible distributions of stresses and rates of stresses.

Using the expressions (2.2) and (2.3), we transform the integral (2.1) into the form

\[ \int_0^T \dot{\epsilon}_{ij} \dot{\sigma}_{ij} \, dv = \int_0^T \dot{\sigma}_{ij}^{(1)} \dot{\epsilon}_{ij} \, dv + \int_0^T \dot{\sigma}_{ij} \dot{\epsilon}_{ij} \, dv. \]

(2.4)

In accordance with the principle of virtual work, the first two of the integrals thus obtained are zero, since each of them represents the work done by a self-equilibrium system of stresses (or stress rates) over kinematically possible strain rates (or strains). Thus we can write

\[ \int_0^T \dot{\epsilon}_{ij} \dot{\sigma}_{ij} \, dv = \int_0^T \dot{\sigma}_{ij}^{(1)} \dot{\epsilon}_{ij} \, dv. \]

(2.5)

We note that the identity (2.4) can be obtained also directly on the basis of the reciprocity theorem of work.

With the expression (2.4) taken into account, the temperature term of Eq. (1.1) assumes the form

\[ \int_0^T \dot{\epsilon}_{ij} \dot{\sigma}_{ij} \, dv = \int_0^T \dot{\sigma}_{ij} \dot{\epsilon}_{ij} \, dv. \]

(2.6)

It is not difficult to see that at the same time we satisfy the condition according to which stationary thermal stresses must not affect the adaptability

\[ \int_0^T \dot{\epsilon}_{ij} \dot{\sigma}_{ij} \, dv = \int_0^T \dot{\sigma}_{ij} \dot{\epsilon}_{ij} \, dv = \int_0^T \dot{\sigma}_{ij} \dot{\epsilon}_{ij} \, dv = 0. \]

(2.7)

since the increments of plastic strains \( \Delta \epsilon_{ij} \) (1.2) form a kinematically possible distribution.

As is known, when the conditions of adaptability are violated, two possibilities exist. In the first case, called the progressive failure, the distribution of increments of the plastic strain per cycle is kinemat-
ically possible and, consequently, it corresponds to a certain mechanism of "instantaneous" failure of the body. An analogy with the situation arising when the load carrying capacity is exhausted is reflected in the similarity between the equations of the kinematic theorems of the theory of adaptability and the equations of the theory of limiting equilibrium [3].

In the second case the plastic strain occurs in a certain region of the body twice per cycle in opposite directions. The total strain per cycle is zero. Therefore, here we can speak of kinematically possible mechanism of failure.

The extension of the theorem of Koiter to the last case is only of certain methodological interest. The results obtained correspond to the well-known criterion, according to which a sign-alternating flow is possible, provided that there exists such a loading program (when intervals of variation of the external actions are specified) for which the variation of the fictitious "elastic" stresses per cycle at least one point of the body exceed the doubled yield point. In fact, this criterion gives the upper limit for the allowable loads, since it presupposes the existence of the corresponding fields of residual stresses. This, however, is not always statically possible. But this result usually coincides with the exact solution, particularly when the plasticity condition of Tresca is used.

In the following the kinematic theorem will be used only with the aim of determining the condition of progressive failure.

3. We confine ourselves to the case where an external load, proportional to a general parameter, acts on the body, and where the temperature field varies cyclically. We proceed from the fact that under these conditions the most unfavorable loading program for the load carrying capacity is thermal change occurring under constant (maximum is absolute value) external load (exceptions are in fact possible here). Using this loading program and denoting the maximum values of the body and surface forces by $X_0^i$ and $p_0^i$, we transform the first two terms of Eq. (1.1) into the form

$$
\int_0^T dt \int \left( \sum_{i,j} \sigma_{ij} \varepsilon^{ij} \right) dv = \int \left( \sum_{i,j} \sigma_{ij} \varepsilon^{ij} \right) dv - \int_0^T \int \sum_{i,j} \sigma_{ij} \varepsilon^{ij} dv dt = \int_0^T \int \sum_{i,j} \sigma_{ij} \varepsilon^{ij} dv dt + \int_0^T \int \sum_{i,j} \sigma_{ij} \varepsilon^{ij} dv dt = \int_0^T \int \sum_{i,j} \sigma_{ij} \varepsilon^{ij} dv dt.
$$

where $\Delta u_{ij} = \int_0^T \sum_{i,j} \sigma_{ij} \varepsilon^{ij} dv dt$ are the increments of displacements in an admissible cycle of the plastic strain rates connected with the strain increments (1.2) by the relations

$$
\Delta \varepsilon^{ij} = \frac{1}{2} \left( \Delta u_{ij} + \Delta u_{ji} \right).
$$

The action of the temperature field leads to additional (thermal) stresses being superimposed. Since the limiting cycle (the boundary of the region of purely elastic behavior of the body) is considered, the plastic strain rates can be nonzero only at those instants of time, when at the given point of the body the superimposed stresses assume the stationary values corresponding to the boundaries of the interval in which they vary. Considering the mechanism of progressive failure, for which at each point of the body a definite flow state is realized per cycle, we denote the corresponding (single) value of the tensor of the additional stresses by $\sigma_{ij}^*$. It is natural to assume that for a given distribution of velocities it corresponds to the algebraic maximum of the product $\sigma_{ij}^* \varepsilon^{ij}$. Hence we obtain

$$\sigma_{ij}^* \varepsilon^{ij} \geq 0 \quad (3.3)$$

and the expression of the temperature term (2.3), with (1.2) taken into consideration, is transformed into

$$\int_0^T \int \sigma_{ij}^* \varepsilon^{ij} dv dt = \int_0^T \sigma_{ij}^* dv \int \varepsilon^{ij} dv dt = \int \sigma_{ij}^* \Delta \varepsilon^{ij} dv dt. \quad (3.4)$$

Here $V_{al}$ is the volume of the additional loading zone in which the strict inequality (3.3) is realized.

In the unloading zone according to the same condition $\sigma_{ij}^* = 0$, the plastic strain rates can be nonzero only at the instants of time, when thermal stresses are absent.

We note that if by the condition of the problem the thermal stresses do not vanish within the duration of the cycle, then, in accordance with (2.6), we can take their distribution at a certain (arbitrary) instant of time as the "zero."

In the general case of a nonstationary temperature field the stresses $\sigma_{ij}^*$ are not reached simultaneously at different points of the loading zone. To determine them, using the terminology adopted in the structural mechanics, we must lay out the volume diagram of thermal stresses. An exception is the case in which thermal stresses at points of the body vary proportionally to the general parameter (which is possible, for example, for a regular thermal state [4]). Here it is sometimes convenient to use the identity transformation based on the relation (3.2)

$$\int \sigma_{ij}^* \Delta \varepsilon^{ij} dv = \int S_{al} \sigma_{ij}^* \Delta u_{ij} dS, \quad (3.5)$$

where $S_{al}$ is the surface bounding the additional loading zone.

Bearing in mind that the state of flow for the mechanism of failure being considered is defined at each point of the body, and taking into account that the dissipative function $F(\Delta \varepsilon_{ij})$ is linearly connected with plastic strain rates, we transform the right side of Eq. (1.1) into the form

$$\int_0^T \int \frac{\partial F}{\partial \Delta \varepsilon^{ij}} dv dt = \int_0^T \int F(\Delta \varepsilon^{ij}) dv dt = \int F(\Delta \varepsilon^{ij}) dv. \quad (3.6)$$

Thus, as applied to the conditions considered here, with the expressions (3.1), (3.4) and (3.6) taken into account, Eq. (1.1) assumes the form

$$\int_0^T \int \left( X^* \Delta u_{ij} dv + \int_0^T \int p^* \Delta \varepsilon^{ij} dS + \int_0^T \int \sigma_{ij}^* \Delta \varepsilon^{ij} dv dt \right) = \int_0^T F(\Delta \varepsilon_{ij}) dv. \quad (3.7)$$

We note that for $\sigma_{ij}^* = 0$ Eq. (3.7) is transformed into the corresponding equation of the theory of limit-