THE AXISYMMETRIC PROBLEM OF THE THEORY OF ELASTICITY FOR A THICK-WALLED CYLINDER OF FINITE LENGTH

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Extensive literature, a review of which is contained in the paper by L. Abramyan and A. Ya. Aleksandrov [1], is devoted to the axisymmetric problem of elasticity for a hollow cylinder. However, the question about the complete and exact satisfaction of the boundary conditions on the surfaces of the cylinder remains open.

In the given paper the problem concerning the satisfaction of the boundary conditions in terms of the stresses is reduced to the solution of infinite systems of linear algebraic equations. It is established that as the number increases, the unknowns tend to a constant. This enabled the method of reduction to be considerably improved and the approximate values to be found for all unknowns. The discussion is illustrated by a numerical example.


In considering the problem concerning the stress state of a cylinder, we can considerably simplify the calculations by separating the symmetric and antisymmetric parts of the stress state. We shall consider in detail the case of stress distribution that is symmetric about the middle surface. At the same time the general boundary problem for a hollow cylinder can be reduced to specifying the nonzero normal stresses either at the end of the cylinder or on its side surfaces.

In the following we shall assume that the nonzero normal stresses are on the side surfaces, i.e., we solve the following boundary value problem:

\[
\begin{align*}
\frac{1}{2G} \sigma_r (r = 1) &= \psi_1 (z); \\
\frac{1}{2G} \sigma_r (r = \epsilon) &= \psi_2 (z); \\
\frac{1}{2G} \tau_{\theta r} (r = \epsilon) &= 0; \\
\frac{1}{2G} \sigma_z (z = \pm \delta) &= 0;
\end{align*}
\]

The outside radius of the cylinder is taken as the characteristic dimension in introducing the dimensionless cylindrical coordinates \( r \) and \( z \). The general case of the boundary value problem differs from the one considered here only by more cumbersome free terms in the infinite systems of linear algebraic equations.

For the stresses (displacements) given on the surfaces of the cylinder, the solution of the Lamé equations must contain two arbitrary functions within the annulus \( \epsilon < r < 1 \) and a pair of arbitrary functions in the interval \( -\delta < z < \delta \) for \( r = \epsilon, r = 1 \). The most convenient form of representing the arbitrary functions on a finite interval is the form of a series of a complete orthogonal system. In connection with this, the desired components of the vector of elastic displacement are represented in the form

\[
\begin{align*}
A_0 + B_0 - \sum_{j=1}^{\infty} \left[ C_j \sin \lambda_j \delta + \frac{3m - 4}{2m \lambda_j} C_j \sin \lambda_j \delta + C_j \sin \lambda_j \delta \right] N_j (\lambda_j \delta) + \\
&+ \sum_{m=1}^{\infty} \left[ \Gamma_1 (\lambda_m) + C_{1m} \sin \lambda_m \delta \right] \cos \lambda_m \delta;
\end{align*}
\]

This enables us to choose the non-self-equilibrium components in the normal stresses. The second part (summation with respect to \( j \)) constitutes the solution for an infinite elastic layer with a "periodic" stress (displacement) distribution on the boundary. This part of the solution includes two arbitrary functions in the annulus \( \epsilon < r < 1 \); they are represented by the series \( N_0 (\lambda_j \delta) \) and \( N_1 (\lambda_j \delta) \). The third part (summation with respect to \( n \)) is the solution for an hollow cylinder, and contains pairs of arbitrary functions on the side surfaces; these functions are represented in the form of series of \( \cos k_n \delta z \) and \( \sin k_n \delta z \).

Hence it follows that the functional arbitrariness in the solution (1.2) of the Lamé equations is sufficient for satisfying the arbitrary boundary conditions, in terms of stresses or displacements, on the surfaces of a hollow cylinder of finite length.

The solution of the Lamé equations with a sufficient degree of arbitrariness can also be constructed without periodic continuation of the boundary conditions into infinite regions. Here the series in (1.2) are replaced by integrals, and a system of integral equa-
The expressions for the components of the stress tensor are found from (1.2) by means of the relationships of Hooke's law:

\[ \sigma_r = \frac{1}{2G} \left( \frac{mB_0 + D_0}{m - 2} - A_0 \right) + \]
\[ + \sum_{j=1}^{\infty} \left( A_j \cosh \lambda_j z + \frac{3m - 4}{\lambda_j^2} C_j \cosh \lambda_j z + C_j \sinh \lambda_j z \right) \times \]
\[ \times \frac{N_1 (b_j r)}{r} - \sum_{j=1}^{\infty} \left( A_j \sinh \lambda_j z + \right. \]
\[ + \frac{3m-2}{m} C_j \sinh \lambda_j z + C_j \sinh \lambda_j z \lambda_j z \right) N_0 (b_j r) + \]
\[ + \sum_{n=1}^{\infty} \left[ - \frac{1}{r} \Gamma_1 (b_n r) + k_n \Gamma_0 (b_n r) + k_n Q_1 (b_n r) + \right. \]
\[ + \frac{1}{b_n r} \left( - \frac{1}{m} Q_1 (b_n r) - \frac{3m-2}{m} Q_0 (b_n r) \right) \times \]
\[ \times \frac{N_1 (b_n r)}{r} \right] \cos k_n z; \]
\[ \sigma_\theta = \frac{1}{2G} \frac{mB_0 + D_0}{m - 2} + A_0 \]
\[ - \sum_{j=1}^{\infty} \left( A_j \cosh \lambda_j z + \frac{3m-4}{\lambda_j^2} C_j \cosh \lambda_j z + \right. \]
\[ + \frac{3m-2}{m} C_j \cosh \lambda_j z + C_j \cosh \lambda_j z \right) N_0 (b_j r) - \]
\[ - \sum_{j=1}^{\infty} \left[ k_n \Gamma_0 (b_n r) + \frac{2}{m} Q_0 (b_n r) + \right. \]
\[ + k_n Q_1 (b_n r) \times \left[ \lambda_n \cosh \lambda_n z + \right. \]
\[ + \frac{m-1}{m} C_j \cosh \lambda_j z + C_j \cosh \lambda_j z \right] \left. N_0 (b_n r) - \right. \]
\[ - \sum_{n=1}^{\infty} \left[ k_n \Gamma_0 (b_n r) + k_n \Gamma_0 (b_n r) + \right. \]
\[ + k_n Q_1 (b_n r) \times \frac{1}{b_n r} \left[ \lambda_n \cosh \lambda_n z + \right. \]
\[ + \frac{m-1}{m} Q_1 (b_n r) \sin k_n z. \]