APPROXIMATE METHOD OF CONFORMAL MAPPING OF REGIONS WITH CORNER POINTS

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In a number of practical problems it is necessary to map a canonical onto a given region. M. A. Lavrent'ev [1, 2], after working out his well-known variational method of conformal mapping, also gave formulas for mapping similar regions. P. F. Fil'chakov [6-8] has studied conformal mappings using the method of trigonometric interpolation. On the basis of an application of Lavrent'ev's formulas to the method of trigonometric interpolation it has proved possible to obtain a more effective method of conformal mapping of the interior of a circle $|z| = 1$ onto the interior of a region bounded by a smooth simple closed curve, which may be given analytically, graphically or by a discrete series of points. The result is generalized to include a region with a finite number of corner points.

§1. The function that allows the mapping in question to be realized with any degree of accuracy is sought in the form of a polynomial:

$$z = P_m(\zeta) = \sum_{j=1}^{m} C_j \zeta^j,$$  \hspace{1cm} (1.1)

where

$$C_j = A_j + iB_j, \quad P_m(0) = 0, \quad P_m(1) = x_0.$$

In order to determine the coefficients $A_j$ and $B_j$ of the mapping function the circle is divided into $2m$ equal parts and two systems of points— even and odd nodal points.

The coefficients $A_j$ and $B_j$ of the mapping function, constructed on the basis of the even nodal points, are expressed in terms of the coordinates of the even nodal points in the form of the trigonometric polynomials coefficients:

$$A_j^{(+m)} = \frac{1}{m} \sum_{v=1}^{m} x_{2v} \cos j\varphi_{2v} + y_{2v} \sin j\varphi_{2v},$$

$$B_j^{(+m)} = \frac{1}{m} \sum_{v=1}^{m} y_{2v} \cos j\varphi_{2v} - x_{2v} \sin j\varphi_{2v},$$  \hspace{1cm} (1.2)

where

$$\varphi_{2v} = \frac{2\pi v}{m};  \quad j = 1, 2, \ldots, m.$$

The distribution of the nodal points at the boundary of the region is not initially known. We shall consider the iteration process by means of which the even nodal points can be determined with any degree of accuracy.

The images $\tilde{x}_k = x_k + i\tilde{y}_k$ of the odd points are called the exterior nodal points and are given by [7]

$$\tilde{x}_k = -\frac{1}{m} \sum_{n=1}^{m} x_{2n} + \sum_{v=1}^{m} y_{2v} \tilde{y}_{2v-1};$$  \hspace{1cm} (1.3)

where

$$y_{2v-1} = \frac{1}{m} \sum_{n=1}^{m} \sin j\varphi_{2v-1}; \quad \varphi_{2v-1} = \frac{(2v - k)\pi}{m}. \hspace{1cm} (1.4)$$

The deviation $\delta_k$ of the exterior nodal points from the contour of the given region can be used to define more exactly the position of the even nodal points by means of Lavrent'ev's formulas for mapping similar domains:

$$z = \frac{1}{2\pi} \int_0^{2\pi} \delta^* \left[ \frac{1}{2\pi} \int_0^{2\pi} \delta \left( \frac{\varphi}{\varphi_0} \right) \frac{\varphi}{\varphi_0} \cos \varphi \, d\varphi \right] \frac{d\varphi}{2\pi},$$

$$\Delta\psi_{2v} = \frac{1}{2\pi} \int_0^{2\pi} \delta^* \left( \frac{\varphi}{\varphi_0} \right) \frac{\varphi}{\varphi_0} \cos \varphi \, d\varphi,$$  \hspace{1cm} (1.5)

where $\delta^*(\varphi)$ is a function giving the deviation of the unit circle from the contour obtained from the given mapping $P_{+1}(t)$.

In the given case the function may be represented as an interpolation trigonometric polynomial which takes given values at the $2m$ points: at the even nodal points zero, at the odd nodal points $\delta_k = -\delta_k \times \frac{1}{P_m(\zeta_k)}$.

The polynomial constructed by the method described in [3, 4] has the form

$$\delta^*(\varphi) = \sum_{i=1}^{m-1} b_i \sin j\varphi + \sum_{i=0}^{m} a_j \cos j\varphi,$$  \hspace{1cm} (1.6)

Here

$$b_i = \frac{1}{m} \sum_{k=1}^{2m-1} \delta_k^* \sin j\varphi_k; \quad a_j = \frac{1}{m} \sum_{k=1}^{2m-1} \delta_k^* \cos j\varphi_k;$$

$k = 1, 3, 5, \ldots, 2m - 1$; the prime attached to the summation sign means that the two extreme terms of the sum are taken with a weight equal to one half.

*The author is deeply indebted to P. F. Fil'chakov for formulating the problem and taking constant interest in its solution.
Table 1 

<table>
<thead>
<tr>
<th>m</th>
<th>2ν−k</th>
<th>Y_{2ν−k}</th>
<th>Y'_{2ν−k}</th>
<th>\hat{Y}'_{2ν−k}</th>
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<td>7</td>
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</table>

Substituting (1.6) in (1.5) and integrating, we get the quantity Δϕ_{2ν}, which may be treated as the approximate displacement of the even nodal points:

\[ \Delta \phi_{2ν} = \sum_{j=1}^{m} a_j \sin j \pi ν - b_j \cos j \pi ν = \]

\[ = \frac{1}{m} \sum_{j=1}^{m} \sum_{k=1}^{m} \delta_k (\cos j \pi ν \sin j \pi ν_{2ν} - \sin j \pi ν \cos j \pi ν_{2ν}) = \]

\[ = \frac{1}{m} \sum_{j=1}^{m} \sum_{k=1}^{m} \delta_k \sin j \pi ν_{2ν-k} = \sum_{k=1}^{m} \delta_k \nu_{2ν-k} \]  

\[ \nu_{2ν-k} = \frac{1}{m} \sum_{j=1}^{m} \sin j \pi ν_{2ν-j}, \quad \nu_{2ν-k} = \frac{(2ν-k) \pi}{m}. \]  

The quantities (1.4) and (1.9) coincide, since sin m \pi 2ν = 0. The coefficients γ_{2ν-k} are constant for given m and do not depend on the region mapped.

Table 1 gives base values of the coefficients γ_{2ν-k} for m = 4, 8, 16, 32 to eight decimal places.

Now, for determining the modulus of the derivative of the approximating polynomial, constructed on the basis of the even nodal points, at the odd points we get

\[ P'_{2ν-k}(ζ_k) = \left| \frac{1}{m} \sum_{j=1}^{m} j \cos j \pi ν_{2ν-j} \right| \]

The expansion at the boundary points may be represented in the form

\[ |P'_{j+m}(ζ_k)| = \left| \frac{1}{m} \sum_{j=1}^{m} j \cos j \pi ν_{2ν-j} \right| = |P'_{2ν-k}(ζ_k)| = \]

\[ = \sqrt{(P'_{2ν-k})^2 + (P'_{2ν-k})^2}. \]  

Keeping in mind formulas (1.2), we find the coefficients of the derivative

\[ |A|_{j+m} = \frac{1}{m} \sum_{v=1}^{m} x_{2ν} \cos j \pi ν_{2ν} + y_{2ν} \sin j \pi ν_{2ν}; \]

\[ |B|_{j+m} = \frac{1}{m} \sum_{v=1}^{m} y_{2ν} \cos j \pi ν_{2ν} - x_{2ν} \sin j \pi ν_{2ν}. \]

Consequently,

\[ x'_{k} = \frac{1}{m} \sum_{v=1}^{m} \sum_{j=1}^{m} j (x_{2ν} \cos j \pi ν_{2ν} + y_{2ν} \sin j \pi ν_{2ν} \cos j \pi ν_{2ν}) \sin j \pi ν_{2ν} = \]

\[ = \frac{1}{m} \sum_{v=1}^{m} \sum_{j=1}^{m} j x_{2ν} \cos j \pi ν_{2ν-k} + j y_{2ν} \sin j \pi ν_{2ν-k}; \]

\[ x'_{k} = \sum_{v=1}^{m} x_{2ν} \nu_{2ν-k} + y_{2ν} \nu_{2ν-k}. \]  

Similarly,

\[ y'_{k} = \sum_{v=1}^{m} y_{2ν} \nu_{2ν-k} - x_{2ν} \nu_{2ν-k}. \]  

Table 1 also gives base values of the coefficients γ_{2ν-k} and \hat{γ}_{2ν-k} for m = 4, 8, 16, 32 to eight decimal places.