EQUILIBRIUM OF A SUPPORTED RECTANGULAR PLATE LYING ON A CROSSED SET OF BEAMS

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1. Let a plate with cylindrical rigidity $D$ occupy the region $0 \leq x \leq a$, $0 \leq y \leq b$, and beams with bending rigidities $B_1^{(1)}$ and $B_2^{(2)}$ be placed respectively on the straightlines $y = y_1$ and $x = x_j$ ($i = 1, 2, \ldots, n; j = 1, 2, \ldots, m$).

The potential energy of such a system is

$$I(v) = \frac{D}{2} \int_0^a \int_0^b \left[ \left( \frac{\partial v}{\partial x} \right)^2 + \left( \frac{\partial v}{\partial y} \right)^2 + 2\left( \frac{\partial^2 v}{\partial x \partial y} \right)^2 \right] \, dx \, dy + \frac{1}{2} \sum_{i=1}^n B_1^{(1)} \int_0^a \left( \frac{\partial v}{\partial x} \right)^2 \, dx + \frac{1}{2} \sum_{j=1}^m B_2^{(2)} \int_0^b \left( \frac{\partial v}{\partial y} \right)^2 \, dy. \quad (1.1)$$

Here $v(x, y)$ are functions that are continuous together with their first partial derivatives and vanish at the boundary; the above functional is meaningful in terms of these functions.

The set of the functions $v(x, y)$ satisfying the conditions mentioned above will be called a class of admissible functions of the system.

If $P(x, y)$ is the strength of the load applied to the system, then according to the principle of virtual displacements the displacements $u(x, y)$ of the points of the system satisfy the equation

$$L(u, v) = \frac{D}{2} \int_0^a \int_0^b \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) \left( \frac{\partial v}{\partial x} \right) \, dx \, dy + \sum_{i=1}^n B_1^{(1)} \int_0^a \frac{\partial^2 u}{\partial x^2} \frac{\partial v}{\partial x} \, dx + \sum_{j=1}^m B_2^{(2)} \int_0^b \frac{\partial^2 u}{\partial y^2} \frac{\partial v}{\partial y} \, dy \times \frac{\partial^2 v}{\partial y^2} \, \, dy = \int_0^a \int_0^b P(x, y) v(x, y) \, dx \, dy, \quad (1.2)$$

where $v(x, y)$ is any admissible function.

In the present work the solution of the problem is expressed in the form of uniformly converging series of two types, which makes it possible to obtain uniform estimates of the errors in replacing these series by their partial sums.

The first representation is obtained by the method of union of systems, which consists in the following.

Let two independent elastic systems $S_1$ and $S_2$ be given, occupying respectively the regions $D_1$ and $D_2$; $v_1^{(1)}(x)$ and $v_2^{(2)}(x)$ are admissible functions of the systems $S_1$ and $S_2$. We shall assume that a reciprocal, unique and continuous correspondence has been somehow established between the points $\xi_1$ and $\xi_2$ belonging to the regions $E_1 \subset D_1$ and $E_2 \subset D_2$.

The system $S$ constructed from the above two systems is called a joint system if all its admissible displacements at the corresponding points $\xi_1$ and $\xi_2$ coincide.

The union can be obtained by putting constraints on the two independent subsystems. Let $P_1^{(1)}(x)$, $x \in D_1$ $(i = 1, 2, \ldots)$ be a sequence of functions from $L_2$, such that the sequence $P_1^{(1)}(x) \equiv 0$, $x \notin E_1$ is closed.
in the region $E_1$. We construct functions $P^{(2)}_i(x)$ according to the following rule: $P^{(2)}_1(x) \equiv 0$, $x \notin E_2$; $P^{(2)}_i(x) = P^{(1)}_i(x)$, $x \in E_2$. Then from the conditions

$$\int_{E_1} c^{(1)}(x) P^{(1)}_i(x) \, dx - \int_{E_2} c^{(1)}(x) P^{(2)}_i(x) \, dx = 0$$

it follows that

$$\psi^{(1)}_i(x) = \psi^{(2)}_i(x).$$

**Theorem 1.** If the action functions $G_1(x, s)$ and $G_2(x, s)$ of the subsystems $S_1$ and $S_2$ are continuous, the action function $K(x, s)$ of the joint system exists and for $x, s \in D_1$ it is the limit of a uniformly converging sequence of functions

$$K_n(x, s) = G_1(x, s) - \sum_{i=1}^{n} \psi_i(x) \psi_i(s).$$

Here

$$\psi_i(x) = \frac{1}{V A_1 A_{n-1}} \begin{vmatrix} \psi_1^{(1)}(x) & \psi_2^{(1)}(x) & \ldots & \psi_n^{(1)}(x) \\ a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n-1,1} & a_{n-1,2} & \ldots & a_{n-1,n} \end{vmatrix}$$

$$\psi_i(x) = \int_{E_1} G_1(x, s) P^{(1)}_i(s) \, ds,$$

$$A_n = \det \| a_{i,k} \|_{i,k=1}^{n} (n>0), \quad A_0 = 1.$$

Without going into the proof of the theorem, which is given in [3], we show how this result can be used in the problem being investigated here.

We take $S_1$ and $S_2$ as systems, each of which occupies the region $0 \leq x \leq a$, $0 \leq y \leq b$, where respective admissible functions $v^{(1)}(x, y)$ and $v^{(2)}(x, y)$ are all continuous with continuous first partial derivatives and vanish at the boundary of the region; in the region of variation of the functions $v^{(1)}(x, y)$ and $v^{(2)}(x, y)$ the following expressions are valid:

$$I_1(y) = \int_{ \int_{a}^{b} \left[ \frac{\partial^2 v}{\partial x^2} \right]^2 + \left( \frac{\partial v}{\partial x} \right)^2 \right] \, dx \, dy + \frac{1}{2} \sum_{i=1}^{n} B^{(1)}_i \int_{a}^{b} \left[ \frac{\partial^2 v(x, y)}{\partial x^2} \right]^2 \, dx + \frac{1}{2} \sum_{j=1}^{m} B^{(2)}_j \int_{a}^{b} \left[ \frac{\partial^2 v(x, y)}{\partial y^2} \right]^2 \, dy,$$

which are the potential energies of the systems $S_1$ and $S_2$ respectively.

It is obvious that the investigated system is obtained by the union of $S_1$ and $S_2$ at all points of the region $0 \leq x \leq a$, $0 \leq y \leq b$. It is not difficult to show that the action functions of the systems $S_1$ and $S_2$ are continuous and positive kernels.

We put

$$P^{(1)}(x, y) = \psi_{ij}(x) \chi_j(y) C_1(x) C_2(y),$$

where $\psi_i(x)$ and $\chi_j(y)$ are the eigenfunctions of the following boundary-value problems:

$$\psi^{IV} = \mu C_1(x) \psi; \quad \psi(0) = \psi(a) = \psi^{'}(0) = \psi^{''}(a) = 0;$$

$$\chi^{IV} = \mu C_2(y) \chi; \quad \chi(0) = \chi(b) = \chi^{'}(0) = \chi^{''}(b) = 0.$$

The expressions for $C_1(x)$ and $C_2(y)$ are of the form

$$C_1(x) = D + \sum_{i=1}^{m} B^{(1)}_i \delta(x - x_i); \quad C_2(y) = D + \sum_{i=1}^{n} B^{(2)}_i \delta(y - y_i),$$

where $\delta(x)$ is Dirac's $\delta$-function.